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A FIXED POINT OF A GROUP OF NONEXPANSIVE MAPS

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Dae Hyeon Pahk [1] proved a result on a fixed point of a group of nonexpansive linear maps. This note shows that linearity is not necessary.

Let K be a nonempty compact convex subset of a Banach space. Let F be a group of nonexpansive transformations of K. For $x \in K$ let $C(F,x)$ be the smallest closed convex subset of K containing x which is invariant under F. Suppose F had regular orbital diameter, that is, for $x \in K$, either $Fx=\{x\}$ or there exist $y, z \in C(F,x)$ such that $\sup\{\|y-Tz\|:T \in F\}$ is less than the diameter $dC(F,x)$ of $C(F,x)$.

THEOREM. Under the conditions above, F has a fixed point.

PROOF. Let K_1 be minimal with respect to being a nonempty closed convex of K which is invariant under F.

Suppose K_1 has more than one point. Take x_0 in K_1. Then there are y_0 and z_0 in $C(F,x_0)$ with $r_0=\sup\{\|y_0-Tz_0\|=T \in F\} < dC(F,x_0)$. Let $M=\{y \in K_1: \sup\{\|y-Tz_0\|: T \in F\} \leq r_0\}$. Then M is nonempty, closed, convex, and invariant under F since F is a group. Hence, $M=K_1$. Let $N=\{z \in K_1: \|y-z\| \leq r_0$ for all $y \in K_1\}$. N is nonempty, closed, convex, and invariant under F since F is a group. But $N \neq K_1$, contradicting minimality of K_1.

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REFERENCE

[1] Dae Hyeon Pahk, *Common fixed point theorem for some bounded linear operators*, Kyungpook Math. J. 13 (1973), 157—159.