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ON THE COMPOSITION OF RELATIONS IN TOPOLOGICAL SPACES

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1. Introduction

We investigate, in this paper, compositive properties, that is, properties preserved under composition of relations (definitions 2.1, 2.2).

In 2, we show that normality, Lindelof, denseness, T_0, T_1, T_2, regularity, complete regularity and completeness are not compositive. Convexity of relations on E^n as well as total boundedness in uniform spaces is compositive (theorems 2.6, 2.13). Denseness is treated in theorem 2.9.

The composition of two open relations is shown to be open (lemma 3.1). For open relations, first axiom, second axiom, local connectedenss and separability are shown to be compositive (corollary 3.5).

The composition of two closed relations need not be closed (example 4.5); however, if one of the relations is compact, then the composition of two closed relations is closed (theorem 4.2).

Compactness is not compositive (example 5.3); however, the composition of two compact relations is compact if one of the relations is closed (theorem 5.1).

Connectedness is not compositive (example 6.1). Various sufficient conditions are given for composition of two relations to be connected (theorems 6.2, 6.3, 6.4, 6.5, 6.6).

In 7, we give sufficient conditions for the composition of two relations to be T_0, T_1, T_2, regular or second axiom in terms of "small" topologies for each of the relations.

2. Definitions, examples, general theorems

We begin with

DEFINITION 2.1. For relations R and S on a set X, the *composition* of R and S (written $R \circ S$) is defined to be $\{(a,b):(a,x) \in S,\ (x,b) \in R \text{ for some } x \in X\}$.

DEFINITION 2.2. A property P is called *compositive* iff $R \circ S$ has property P when R and S each have property P.

DEFINITION 2.3. A property P is *biproductive* iff $X \times X$ has property P when

X has property P.

A necessary condition for a topological property to be compositive is given in

THEOREM 2.4. *Every compositive topological property is biproductive.*

PROOF. Let P be a non biproductive topological property. Then there exists a topological space X with property P for which $X\times X$ does not have property P. Let $x\in X$, $R=\{x\}\times X$ and $S=X\times\{x\}$. Now R and S are each homeomorphic to X and thus have property P. But $R\circ S=X\times X$ and hence property P is not compositive.

COROLLARY 2.5. *Normality and Lindelof are not compositive properties.*

THEOREM 2.6. *Let $R, S\subset E^n\times E^n$, R and S being convex. Then $R\circ S$ is convex.*

PROOF. Let $(a,b)\in R\circ S$ and $(c,d)\in R\circ S$; then $(a,x)\in S$ and $(x,b)\in R$ for some $x\in X$ and $(c,y)\in S$, $(y,d)\in R$ for some $y\in X$. Let $0\leq t\leq 1$. But $t(a,x)+(1-t)(c,y)\in S$ and $t(x,b)+(1-t)(y,d)\in R$. It follows that $t(a,b)+(1-t)(c,d)\in R\circ S$.

Denseness is not compositive as shown in

EXAMPLE 2.7. Let $R=\{(r,s)\colon r,s \text{ rational}\}$ and $S=\{(a,b)\colon a,b \text{ irrational}\}$. R and S are each dense in E^2, but $R\circ S=\phi$.

LEMMA 2.8. *Let R and S be relations on a topological space X for which $R^{-1}[x]$ is dense for $x\in P_2[R]$ and S is open. Then $R\circ S=P_1[S]\times P_2[R]$.*

PROOF. Clearly, $R\circ S\subset P_1[S]\times P_2[R]$. Let then $(a,b)\in P_1[S]\times P_2[R]$. Now $S[a]\neq\phi$ and is open and $R^{-1}[b]$ is dense; let $x\in S[a]\cap R^{-1}[b]$. Then $(a,x)\in S$ and $(x,b)\in R$; $(a,b)\in R\circ S$.

THEOREM 2.9. *Let $R^{-1}[x]$ be dense for each $x\in P_2[R]$ and let S be an open relation on X. Then $R\circ S$ is dense iff $P_1[S]$ and $P_2[R]$ are each dense.*

PROOF. By lemma 2.8, $R\circ S=P_1[S]\times P_2[R]$. Then $c(R\circ S)=X\times X$ iff $cP_1[S]\times cP_2[R]=X\times X$ iff $cP_1[S]=X=cP_2[R]$.

T_1, T_2, regularity, complete regularity are not compositive as shown in

EXAMPLE 2.10. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi,\{b\},\{c\},\{b,c\},X\}$; let $S=\{(a,c),(b,b)\}$ and $R=\{(b,b),(c,a)\}$. Then R and S are each discrete spaces, but $R\circ S=\{(a,a),(b,b)\}$ which is neither T_1 nor regular.

T_0 is not compositive as shown in

EXAMPLE 2.11. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi, \{c\}, X\}$. If $R=\{(c,a),(b,b)\}$ and $S=\{(a,c),(b,b)\}$, then R and S are each T_0-spaces, but $R\circ S=\{(a,a),(b,b)\}$ which is not a T_0-space.

EXAMPLE 2.12. Let X be the reals with the usual metric. If $R=\left\{\left(x,\frac{1}{x}\right):x>0\right\}$ and $X=S$, then R and S are complete spaces, but $R\circ S=\{(x,x):x>0\}$ which is not complete.

THEOREM 2.13. *Let $(X,\mathscr{U})$ be a uniform space; let R and S be totally bounded relations on X (relative to $(X,\mathscr{U})\times(X,\mathscr{U})$). Then $R\circ S$ is totally bounded.*

PROOF. $R\circ S\subset P_1[S]\times P_2[R]$ and $P_1[S]$ and $P_2[R]$ are each totally bounded since projection maps are uniformly continuous. But products of totally bounded spaces are totally bounded and subspaces of totally bounded spaces are totally bounded. It follows that $R\circ S$ is totally bounded.

3. Open relations

LEMMA 3.1. *Let R and S be open relations on a topological space X. Then $R\circ S$ is open.*

PROOF. $R\circ S=\bigcup\{S^{-1}[x]\times R[x]: x\in X\}$. But S^{-1} is open and sections of open sets are open.

COROLLARY 3.2. *Let R and S be relations on a topological space X. Then* $\mathrm{Int}R\circ\mathrm{Int}S\subset\mathrm{Int}(R\circ S)$.

EXAMPLE 3.3. Let X be the reals, $R=\{(0,y):y\in X\}$ and $S=\{(x,0): x\in X\}$. $\mathrm{Int}(R\circ S)=X\times X\not\subset\phi=\mathrm{Int}R\circ\mathrm{Int}S$.

THEOREM 3.4. *Let P be a property invariant under open continuous surjections, productive and open-hereditary. If R and S are open relations on X and have property P, then $R\circ S$ has property P.*

PROOF. By lemma 3.1, $R\circ S$ is open in $P_1[S]\times P_2[R]$ and since $P_1|S$ and $P_2|R$ are open maps, $P_1[S]$ and $P_2[R]$ have property P, and thus, so does $P_1[S]\times P_2[R]$. The theorem follows.

COROLLARY 3.5. *Let R and S be open relations on X. If R and S are each first axiom (second axiom, separable, locally connected), then $R\circ S$ is first axiom* *(second axiom, separable, locally connected).*

4. Closed relations

LEMMA 4.1. *Let R and S be relations on a topological space X. If $P_2[S]$ or $P_1[R]$ is compact, then* $c(R\circ S) \subset c(R)\circ c(S)$.

PROOF. Let $(x,y) \in c(R\circ S)$; then $(x,y)=\lim\{(x_d, y_d):d \in D\}$ where $(x_d, y_d) \in R\circ S$ and $d \in D$, a directed set. Then for each $d \in D$, there exists a $t_d \in X$ such that $(x_d, t_d) \in S$ and $(t_d, y_d) \in R$. Case 1. $P_2[S]$ is compact. Then without loss of generality, we may assume that $\lim\{t_d:d \in D\}=t$. Then $\lim\{(x_d, t_d):d \in D\} = (x,t) \in c(S)$; also $\lim\{(t_d, y_d): d \in D\}=(t,y) \in cR$. Hence $(x,y) \in c(R)\circ c(S)$. Case 2. $P_1[R]$ is compact. The proof is similar to case 1.

THEOREM 4.2. *Let R and S be closed relations on a topological space X. If either R or S is compact, then $R\circ S$ is closed.*

PROOF. $P_2[S]$ or $P_1[R]$ is compact; hence $c(R\circ S) \subset c(R)\circ c(S)=R\circ S$ by lemma 4.1.

COROLLARY 4.3. *Let R and S be closed relations on a compact space X. Then $R\circ S$ is closed.*

COROLLARY 4.4. *Let R and S be $\mathscr{F}_\sigma$-relations on a topological space X. If R or S is compact, then $R\circ S$ is an $\mathscr{F}_\sigma$.*

PROOF. $R=\bigcup\{R_i: i \geq 1\}$ and $S=\bigcup\{S_j: j \geq 1\}$ where R_i and S_j are each closed. Then $R\circ S=\bigcup\{R_i\circ S_j: i \geq 1, j \geq 1\}$ and R_i or S_j is compact. Then $R_i\circ S_j$ is closed by theorem 4.2.

EXAMPLE 4.5. Let $R=S=\left\{\left(x, \frac{1}{x}\right): x>0\right\}$ be relations on the set of reals. $R\circ S = \{(x,x):x>0\}$ and $c(R\circ S)=\{(x,x): x \geq 0\} \not\subset R\circ R=c(R)\circ c(S)$.

5. Compactness

THEOREM 5.1. *Let R and S be compact(sequentially compact) relations on a space X and suppose that R is closed or S is closed. Then $R\circ S$ is compact (sequentially compact).*

PROOF. We prove only the compact case. Let $\{(x_d, y_d): d \in D\}$ be a net of points in $R\circ S$. Then for each $d \in D$, there exists a $t_d \in X$ such that $(x_d, t_d) \in S$ and $(t_d, y_d) \in R$. Without loss of generality, we may assume by compactness of

R and S that $\lim(x_d, t_d)=(x, t_1)\in S$ and $\lim(t_d, y_d)=(t_2, y)\in R$. Case 1. S closed. Now $\lim(x_d, t_d)=(x, t_2)$ and hence $(x, t_2)\in S$. Thus $\lim(x_d, y_d)=(x, y)\in R\circ S$. Case 2. R is closed. Modify case 1.

COROLLARY 5.2. *Let X be Hausdorff and R and S σ-compact relations. Then $R\circ S$ is σ-compact.*

EXAMPLE 5.3. Let $X=(0, 2]$ and $\mathscr{T}=\{A: A=X \text{ or } A\cap\left\{2, \frac{1}{2}\right\}=\phi\}$. Let $S=\left\{\left(x, \frac{1}{x}\right): \frac{1}{2}\leq x\leq 1\right\}$; then $\left(\frac{1}{2}, 2\right)\in S$ and hence S is compact. Let $R=\left\{\left(x, \frac{1}{x}\right): 1\leq x<2\right\}\cup\left\{\left(\frac{1}{2}, 2\right)\right\}$. Since $\left(\frac{1}{2}, 2\right)\in R$, it follows that R is compact. But $R\circ S=\{(x, x): \frac{1}{2}<x\leq 1\}$, an infinite discrete space and hence not compact. Note that $\left(2, \frac{1}{2}\right)\in c(R)\cap c(S)$, but $\left(2, \frac{1}{2}\right)\notin R$, $\left(2, \frac{1}{2}\right)\notin S$.

6. Connectedness

EXAMPLE 6.1. Connectedness is non compositive. Let $X=[0, 1]$ with the usual topology. Let $S=\left\{\left(\frac{1}{4}, \frac{3}{4}\right)\right\}$; let $R=\left\{\left(x, \frac{1}{2}\right): \frac{3}{4}\leq x\leq \frac{7}{8}\right\}\cup\left\{\left(\frac{7}{8}, y\right): \frac{1}{4}\leq y\leq \frac{1}{2}\right\}\cup\left\{\left(x, \frac{1}{4}\right): \frac{3}{4}\leq x\leq \frac{7}{8}\right\}$. Then S and R are each connected, but $R\circ S=\left\{\left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right\}$, a disconnected set.

THEOREM 6.2. *Let R and S be open relations on a space X. If $S[x]$ is connected for each $x\in X$ and $R[A]$ is connected when A is connected, then $R\circ S$ is connected iff $P_1[R\circ S]$ is connected.*

PROOF. If $R\circ S$ is connected, then $P_1[R\circ S]$ is connected since P_1 is continuous. Conversely, suppose that $P_1[R\circ S]$ is connected and that $R\circ S$ is disconnected. By lemma 3.1, $R\circ S$ is open and hence there eixst nonempty disjoint open sets A and B for which $R\circ S=A\cup B$. Now for each $x\in X$, $R\circ S[x]=A[x]\cup B[x]$ and since $A\cap B=\phi$, it follows that $A[x]\cap B[x]=\phi$. But $A[x]$ and $B[x]$ are open and $R[S[x]]$ is connected; thus $A[x]=\phi$ or $B[x]=\phi$ for each $x\in X$. It follows then that $P_1[A]\cap P_1[B]=\phi$. Thus $P_1[R\circ S]=P_1[A]\cup P_1[B]$, $P_1[A]$ and $P_1[B]$ being non empty, open and disjoint. Hence $P_1[R\circ S]$ is disconnected, a contradiction.

THEOREM 6.3. *Let X be compact and Hausdorff. Suppose R and S are closed relations for which $S[x]$ is connected for each $x\in X$ and $R[A]$ is connected when A is connected. Then $R\circ S$ is connected iff $P_1[R\circ S]$ is connected.*

PROOF. Modify the proof of theorem 6.2 using the facts that $R\circ S$ is closed by corollary 4.3, P_1 is a closed map since X is compact Hausdorff and point sections of closed sets are closed.

THEOREM 6.4. *Let R be a connected relation on X and $S\supset\Delta$ (Δ denoting the diagonal); if $S^{-1}[x]$ is connected for each $x\in P_1[R]$, then $R\circ S$ is connected.*

PROOF. We made use of the identity $R\circ S=\bigcup\{S^{-1}[x]\times\{y\}: (x,y)\in R\}$. Now $S^{-1}[x]\times\{y\}$ and hence $S^{-1}[x]\times\{y\}\cup R$ is connected when $(x,y)\in R$. But $R\subset R\circ S$ and hence $R\circ S=\bigcup\{S^{-1}[x]\times\{y\}\cup R: (x,y)\in R\}$, a connected set.

THEOREM 6.5. *Let S be connected and $R\supset\Delta$. If $R[y]$ is connected for all $y\in P_2[S]$, then $R\circ S$ is connected.*

PROOF. Here, we use $R\circ S=\bigcup\{\{x\}\times R[y]: (x,y)\in S\}$. Then $\{x\}\times R[y]\cup S$ is connected when $(x,y)\in S$ and $S\subset R\circ S$. Thus $R\circ S=\bigcup\{\{x\}\times R[y]\cup S: (x,y)\in S\}$, a connected set.

THEOREM 6.6. *Let S be a connected relation on X and R a relation on X such that $R[y]$ is connected when $y\in P_2[S]$. If $R=R^{-1}$ and $S\subset R\circ S$, then $R\circ S$ is connected.*

PROOF. $R\circ S=\bigcup\{\{x\}\times R[y]\cup S:(x,y)\in S\}$. It suffices to show that $(\{x\}\times R[y])\cap S\neq\phi$ when $(x,y)\in S$. Now $(x,y)\in S\subset R\circ S$ implies that $(x,t)\in S$, $(t,y)\in R$ for some t. Hence $t\in R^{-1}[y]=R[y]$ and thus $(x,t)\in S\cap(\{x\}\times R[y])$.

7. Small topologies on R and S

DEFINITION 7.1. For a topological space $(X,\mathscr{T})$, let $X\times\mathscr{T}=\{X\times O: O\in\mathscr{T}\}$ and $\mathscr{T}\times X=\{O\times X: O\in\mathscr{T}\}$. $\mathscr{T}\times\mathscr{T}$ denotes the usual product topology on $X\times X$.

LEMMA 7.2. *Let $A,B\subset X$ and $R,S\subset X\times X$. Then $A\times B\cap R\circ S=((X\times B)\cap R)\circ((A\times X)\cap S)$.*

THEOREM 7.3. *Let R and S be relations on a topological space $(X,\mathscr{T})$ and suppose that $(\mathscr{T}\times X)\cap S$ and $(X\times\mathscr{T})\cap R$ are Hausdorff topologies.*
Then $\mathscr{T}\times\mathscr{T}\cap R\circ S$ is a Hausdorff topology.

PROOF. Let $(a,b)\neq(c,d)$ in $R\circ S$. Case 1. $a\neq c$. Now $(a,x)\in S$ and $(x,b)\in R$ and $(c,y)\in S$ and $(y,d)\in R$ for some x and y. But $(a,x)\neq(c,y)$ in S and hence there exist O_1 and O_2 in $\mathscr{T}$ for which $(a,x)\in(O_1\times X)\cap S$, $(c,y)\in(O_2\times X)\cap S$

and $((O_1 \cap O_2)\times X)\cap S=\phi$. Then $(a,b)\in(O_1\times X)\cap R\circ S$ and $(c,d)\in(O_2\times X)\cap R\circ S$. But $(O_1\times X)\cap(R\circ S)\cap(O_2\times X)\cap(R\circ S)=((O_1\cap O_2)\times X)\cap R\circ S=((X\times X)\cap R)\circ(((O_1\cap O_2)\times X)\cap S)$ (by lemma 7.2) $=R\circ\phi=\phi$. Case 2. $b\neq d$. The proof is similar.

THEOREM 7.4. *Let R and S be relations on X, a topological space. If* $(\mathcal{T}\times X)\cap S$ *and* $(X\times\mathcal{T})\cap R$ *are* T_1*-topologies, then* $(\mathcal{T}\times\mathcal{T})\cap R\circ S$ *is a* T_1*-topology.*

PROOF. Let $(a,b)\neq(c,d)$ in $R\circ S$. Then $(a,x)\in S$, $(x,b)\in R$ and $(c,y)\in S$, $(y,d)\in R$ for some x and y. Case 1. $b\neq d$. Then $(x,b)\neq(y,d)$ in R and hence there exists an $O\in\mathcal{T}$ such that $(x,b)\in(X,O)\cap R$ and $(y,d)\notin(X\times O)\cap R$. Then $d\notin O$. Now $(a,b)\in(X\times O)\cap R\circ S$ and $(c,d)\notin(X\times O)\cap R\circ S$.

Case 2. $a\neq c$. Similar.

THEOREM 7.5. *Let R and S be relations on X, a topological space. If* $(\mathcal{T}\times X)\cap S$ *and* $(X\times\mathcal{T})\cap R$ *are* T_0*-topologies, then* $(\mathcal{T}\times\mathcal{T})\cap R\circ S$ *is a* T_0*-topology.* We omit the proof.

THEOREM 7.6. *Let R and S be relations on a topological space X. If* $(\mathcal{T}\times X)\cap S$ *and* $(X\times\mathcal{T})\cap R$ *are regular topologies, then* $(\mathcal{T}\times\mathcal{T})\cap R\circ S$ *is a regular topology.*

PROOF. Let $(a,b)\in O_1\times O_2\cap R\circ S=(X\times O_2)\cap R\circ(O_1\times X)\cap S$ (by lemma 7.2). Then there exists an $x\in X$ such that $(a,x)\in(O_1\times X)\cap S$ and $(x,b)\in(X\times O_2)\cap R$. Since $(\mathcal{T}\times X)\cap S$ is regular, there exist U_1 a d V_1 in $\mathcal{T}$ such that $(a,x)\in(U_1\times X)\cap S\subset(O_1\times X)\cap S$ and $(\mathscr{C}O_1\times X)\cap S\subset(V_1\times X)\cap S$ with $((U_1\cap V_1)\times X)\cap S=\phi$. Likewise, there eixst U_2 and V_2 in $\mathcal{T}$ such that $(x,b)\in(X\times U_2)\cap R\subset(X\times O_2)\cap R$ and $(X\times\mathscr{C}O_2)\cap R\subset(X\times V_2)\cap R$ with $(X\times(U_2\cap V_2))\cap R=\phi$. Now $(a,b)\in((X\times U_2)\cap R)\circ((U_1\times X)\cap S)\subset(X\times O_2)\cap R\circ(O_1\times X)\cap S$. By lemma 7.2, $(a,b)\in(U_1\times U_2)\cap R\circ S\subset(O_1\times O_2)\cap R\circ S$. Also, $\mathscr{C}[O_1\times O_2]\cap(R\circ S)=((\mathscr{C}O_1\times X)\cap(R\circ S))\cup((X\times\mathscr{C}O_2)\cap(R\circ S))=R\circ((\mathscr{C}O_1\times X)\cap S)\cup((X\times\mathscr{C}O_2)\cap R)\circ S\subset R\circ((V_1\times X)\cap S)\cup((X\times V_2)\cap R)\circ S=(V_1\times X)\cap(R\circ S)\cup(X\times V_2)\cap(R\circ S)$ (by lemma 7.2) $=(V_1\times X\cup X\times V_2)\cap(R\circ S)$. But $(U_1\times U_2)\cap(R\circ S)$ and $(V_1\times X\cup X\times V_2)\cap(R\circ S)$ are disjoint as the reader can verify.

THEOREM 7.7. *Let R and S be relations on X, a topological space and suppose that* $(X\times\mathcal{T})\cap R$ *and* $(\mathcal{T}\times X)\cap S$ *are second axiom topologies. Then* $(\mathcal{T}\times\mathcal{T})\cap(R\circ S)$ *is a second axiom topology.*

PROOF. Let $\{(X\times U_i)\cap R : i\geqq 1\}$ be a base for $(X\times\mathscr{T})\cap R$ and $\{(V_j\times X)\cap S : j\geqq 1\}$ be a base for $(\mathscr{T}\times X)\cap S$. Then $\{(V_j\times U_i)\cap(R\circ S) : i,j\geqq 1\}$ is a base for $(\mathscr{T}\times\mathscr{T})\cap(R\circ S)$. To see this, let $(a,b)\in(V\times U)\cap(R\circ S)$, U and V in $\mathscr{T}$. Then there exists an $x\in X$ for which $(a,x)\in(V\times X)\cap S$ and $(x,b)\in(X\times U)\cap R$. Then $(a,x)\in(V_j\times X)\cap S\subset(V\times X)\cap S$ and $(x,b)\in(X\times U_i)\cap R\subset(X\times U)\cap R$ for some j and i. Then $(a,b)\in(V_j\times U_i)\cap(R\circ S)=((X\times U_i)\cap R)\circ((V_j\times X)\cap S)\subset((X\times U)\cap R)\circ((V\times X)\cap S)=(V\times U)\cap(R\circ S)$.

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