Kyungpook Math. J.
Volume 15, Number 2
December, 1975

POINTWISE PERIODIC SEMIGROUPS AND FULL IDEALS

By Younki Chae

In this paper, a new proof of the structure theorem of pointwise periodic semigroups on an arc is given. If we talk about an ideal A of a semigroup S, it may happen $SA = A = AS$ in many cases. Pointwise periodic semigroups serve as an example of semigroup in which every ideal has this property. The aim of the second half of this note is to find a necessary and sufficient condition that every ideal of a semigroup has such property. The author found that such an ideal property is closely related with regular semigroups. However, still this problem remains open.

1. Pointwise periodic semigroups

A topological semigroup is a Hausdorff space with a continuous associative multiplication, denoted by juxtaposition [2], [3]. Throughout, a semigroup will mean a topological semigroup. An arc is a continuum with exactly two non-cutpoints. It is well known that any arc admits a total order and has one non-cutpoint as a least element and the other non-cutpoint as a greatest element [6]. It is supposed that an arc to have such a total order on it. We will denote an arc with end points a and b, $a<b$, by $[a, b]$ and if $x, y \in [a, b]$, $x<y$, then

$$[x, y] = \{t \mid x \leq t \leq y\}, \quad \langle x, y\rangle = \{t \mid x < t < y\}.$$

A standard thread is a semigroup on an arc in which the greatest element is an identity and the least element is a zero. The real unit interval $[0, 1]$ under the ordinary multiplication is called the real thread, and the real inverval $\left[\frac{1}{2}, 1\right]$ under the multiplication

$$xy = \max\left\{\frac{1}{2}, \text{ ordinary product of } x \text{ and } y\right\}$$

is called the nil thread.

The following lemma gives the structure of standard threads, and will be found in [2].

This work is done under the support of Korean Traders Scholarship Foundation.

LEMMA 1.1. *Let S be a standard thread and let E be the set of all idempotents of S. If* $\langle e, f\rangle$ *is a component of* $S\backslash E$*, then* $[e, f]$ *is iseomorphic to either the real thread or the nil thread.*

DEFINITION 1.2. An element x of a semigroup S is termed *periodic* if and only if $x^{n+1}=x$ for some positive integer n. Also, S is *pointwise periodic* if and only if each element of S is periodic [4], [5], [7].

LEMMA 1.3. *The homomorphic image of a pointwise periodic semigroup is pointwise periodic.*

PROOF. Let $f: S \longrightarrow T$ be a homomorphism of a pointwise periodic semigroup S onto a semigroup T. For any element t of T, let $s \in S$ such that $f(s)=t$. Since S is pointwise periodic, there is a positive integer n such that $s^{n+1}=s$. Then

$$t^{n+1}=f(s)^{n+1}=f(s^{n+1})=f(s)=t,$$

and hence T is pointwise periodic.

LEMMA 1.4. *Let be a semigroup with a zero. If* f*:* $S \longrightarrow T$ *is a homomorphism of onto a semigroup* T $a \in S$ *and if is a nilpotent element, then* $f(a)$ *is a nilpotent element of* T.

PROOF. Since a is a nilpotent element, there is a positive integer n with $a^n=0$. Then

$$f(a)^n=f(a^n)=f(0)=0,$$

that is, $f(a)$ is a nilpotent element of T.

THEOREM 1.5. *Every pointwise periodic standard thread is a semilattice.*

PROOF. Let S be a pointwise periodic standard thread and let $\langle e, f\rangle$ be a component of $S\backslash E$. By lemma 1.1, $[e, f]$ is iseomorphic to either the real thread or the nil thread. Let $a \in \langle e, f\rangle$. If $[e, f]$ is iseomorphic to the real thread, then $a^{n+1} \neq a$ for any positive integer n. This contradicts lemma 1.3. If $[e, f]$ is iseomorphic to the nil thread, then a is a nilpotent element by lemma 1.4. Suppose $a^m=0$ and $a^{p+1}=a$ with m and p positive integers. We may assume $m< p+1$ without loss of generality. Thus,

$$a=a^{p+1}=a^m a^{p+1-m}=0a^{p+1-m}=0$$

which is a contradiction. Therefore E is dense in S. Since E is closed, one obtain $S = E$. Then the commutativity of standard thread gives S a semilattice.

2. Full ideals of semigroups

DEFINITION 2.1. A subset A of a semigroup S is said to be *full* if and only if $SA = A = AS$.

It is easy to see that every ideal of a semigroup with an identity is full.

THEOREM 2.2. (1) *The minimal ideal of a semigroup is full.*

(2) *The closure of a full ideal in a compact semigroup is full.*

(3) *The union of any collection of full ideals of a semigroup in full.*

PROOF. (1) Let K be the minimal ideal of a semigroup S. Then $SK \subset K$ and $KS \subset K$. Since

$$S(SK) = (SS)K \subset SK, \quad (SK)S = S(KS) \subset SK,$$

SK is an ideal of S. The minimality of K gives

$$SK = K = KS.$$

The continuity of multiplication in S gives the proof of (2). The proof of (3) is clear.

The folowing is a characterization of pointwise periodic semigroups as indicated in [4], [5].

LEMMA 2.3. *In order that a semigroup S be pointwise periodic, it is necessary and sufficient that $A^2 \subset A$ implies $A^2 = A$ for all $A \subset S$.*

THEOREM 2.4. *If a semigroup S is pointwise periodic, then every ideal of S is full.*

PROOF. Let A be an ideal of S. Then $SA \cup AS \subset A$, and $A^2 \subset A$. By lemma 2.3, $A^2 = A$. Since $A = A^2 \subset SA \cap AS$, $SA = A = AS$.

THEOREM 2.5. *Every full ideal of a connected semigroup is connected.*

PROOF. Let A be a full ideal of the connected semigroup S. Then $SA = A = AS$. Since S is connected and since the multiplication in S is continuous, aS and Sa are connected for each $a \subset S$. Now since

$$A = AS = \cup \{ aS \mid a \in A \} \cup Sb,$$
$$aS \cap Sb \neq \phi, \quad \forall a \in S,$$

A is connected.

THEOREM 2.6. *Every closed ideal of a compact semigroup S contains a full ideal of S.*

PROOF. Let A be a closed ideal of S. Then

$$A \supset A^2 \supset \cdots \supset A^n \supset \cdots.$$

Since $\{A^n \mid n=1,2,\cdots\}$ is a tower of closed sets in compact S,

$$B = \bigcap \{A^n \mid n=1,2\cdots\} \neq \phi$$

and

$$B^2 = (\bigcap \{A^n \mid n=1,2,\cdots\})^2 = \bigcap \{A^n \mid n=2,3,\cdots\} = B.$$

Again, since $\{S\}$ is a twor,

$$SB = \bigcap \{SA^n \mid n=1,2,\cdots\} \subset \bigcap \{A^n \mid n=1,2,\cdots\} = B.$$

Hence B is a full ideal of S contained in A.

DEFINITION 2.7. An element x of a semigroup S is said to be *regular* if and only if $x \in xSx$, i.e., there is an element y of S such that $xyx=x$. S is termed *regular* if and only if every element of S is regular.

For example, pointwise periodic semigroups, inverse semigroups, and Cliffordean are all regular semigroups.

LEMMA 2.8. *A semigroup S is regular if and only if $A \cap B = AB$ for every right ideal A and every left ideal B of S* [1].

THEOREM 2.9. *Every ideal of a regular semigroup S is full.*

PROOF. Let A be an ideal of S. By lemma 2.8, $A^2 = A$. Therefore

$$SA \subset A = A^2 \subset SA, \quad AS \subset A = A^2 \subset AS.$$

Hence A is full.

COROLLARY 2.10. *Every ideal of a connected regular semigroup is connected.*

Kyungpook University

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