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THE OPERATOR $T_{k,q}$ AND A GENERALIZATION OF CERTAIN CLASSICAL POLYNOMIALS

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1. Introduction

Employing the operator x^2D where $D=\frac{d}{dx}$ Chak [4] defined the generalized Laguerre polynomials by means of

$$L_n^{(\alpha)}(x)=x^{-\alpha-n-1}e^x\ (x^2D)^n\ (x^{\alpha+1}\ e^{-x})\ . \tag{1.1}$$

Later, Al-Salam [1] characterized these polynomials in terms of the operator $\theta=x(1+xD)$ and proved that

$$\theta^n x^\alpha e^{-x}=x^{\alpha+n}\ e^{-x} n!\ L_n^{(\alpha)}(x)\ . \tag{1.2}$$

Recently, Mittal [7] observed that relations (1.1) and (1.2) can in fact be derived from a more general operational representation. To this end he considered the operator $T_k=x(k+xD)$, k being a constant and showed that polynomial set $\{T_{\nu n}^{(\alpha)}(x);\ n=0,\ 1,\ 2,\ \cdots\}$, [6],

$$T_{\nu n}^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha}e^{p_\nu(x)}D^n(x^{\alpha+n}\ e^{-p_\nu(x)}), \tag{1.3}$$

$p_\nu(x)$ being a polynomial in x of degree r, admit the relationship

$$T_{\nu n}^{(\alpha+k-1)}(x)=\frac{1}{n!}\ x^{-\alpha-n}\ e^{p_\nu(x)}T_k^n[x^\alpha e^{-p_\nu(x)}] \tag{1.4}$$

in terms of the operator T_k.

The question naturally arises, if these aforementioned characterizations can be unified. This led us to define the operator $T_{k,q}\equiv x^q(k+xD)$ and the introduction of the polynomial set $\{M_{\nu n}^{(\alpha)}(x,k,q);\ n=0,\ 1,\ 2,\ \cdots\}$ in the form

$$M_{\nu n}^{(\alpha)}(x,k,q)=\frac{1}{n!}x^{-\alpha-nq}\ e^{p_\nu(x)}\,T_{k,q}^n\ [x^\alpha e^{-p_\nu(x)}] \tag{1.5}$$

where $p_\nu(x)$ is a polynomial in x of degree r, k and q are constants.
In what follows, it shall be understood that

$$M_{-\nu n}^{(\alpha)}(x,k,q)=\frac{1}{n!}\,x^{-\alpha-nq}\,e^{-p_\nu(x)}\,T_{k,q}^{n}\,[x^{\alpha}\,e^{p_\nu(x)}]\ . \tag{1.6}$$

For the sake of simplicity, in the present paper, we shall, however, confine ourselves to the case when $k=0$, viz, the polynomials

$$M_{\nu n}^{(\alpha)}(x,q)=\frac{1}{n!}x^{-\alpha-nq}\,e^{p_\nu(x)}\,T_{q}^{n}[x^{\alpha}e^{-p_\nu(x)}],\, T_{q}=x^{q+1}D \tag{1.7}$$

which are connected with the polynomials $M_{\nu n}^{(\alpha)}(x,k,q)$ by the relation

$$M_{\nu n}^{(\alpha+k)}(x,q)=M_{\nu n}^{(\alpha)}(x,k,q). \tag{1.8}$$

The study of the polynomials $M_{\nu n}^{(\alpha)}(x,k,q)$ where $p_\nu(x)$ will be replaced by px^r, for obtaining many nicer properties from the point of view of their utility, will form the subject matter of a subsequent communication.

2. The operator $T_{k,q}$ and its properties

We define the operator $T_{k,q}\equiv x^{q}(k+xD)$ and as a first consequence, note that

$$T_{k,q}^{n}(x^{\alpha+m})=q^{n}\left(\frac{\alpha+m+k}{q}\right)_{n}x^{\alpha+m+nq} \tag{2.1}$$

where m is an integer, n a non-negative integer and α is arbitrary.

Further, by induction

$$T_{k,q}^{n}=x^{nq}\prod_{j=0}^{n-1}(\delta+k+jq),\quad \delta\equiv xD \tag{2.2}$$

so that when $k=0$

$$T_{q}^{n}=x^{nq}\prod_{j=0}^{n-1}(\delta+jq)\ . \tag{2.3}$$

Next, assuming that a function $f(x)$ has a Taylor's series expansion, it follows that the operator $T_{k,q}$ satisfies the following formal rules:

$$F(T_{k,q})\{x^{\alpha}f(x)\}=x^{\alpha}F\{T_{k,q}+x^{q}\alpha\}f(x) \tag{2.4}$$

$$F(T_{k,q})\{e^{g(x)}f(x)\}=e^{g(x)}F[T_{k,q}+x^{q+1}g'(x)]f(x) \tag{2.5}$$

and the analogue of the Leibnitz formula

$$T_{k,q}^{n}(xuv)=x\sum_{m=0}^{n}\binom{n}{m}(T_{k,q}^{n-m}v)\,(T_{1,q}^{m}u),\ T_{1,q}=x^{q}(1+xD)\ . \tag{2.6}$$

For $k=0$ and $x=1$ the above formula reduces to

$$T_q^n(uv)=\sum_{m=0}^{n}\binom{n}{m}(T_q^{n-m}v)(T_q^m u)\ . \tag{2.7}$$

Further, note that (2.6) implies

$$e^{tT_{k,q}}(xuv)=x(e^{tT_{k,q}}v)(e^{tT_{1,q}}u) \tag{2.8}$$

and consequently with the help of (2.1), we have

$$e^{tT_{k,q}}(x^{\alpha+m})=\frac{x^{\alpha+m}}{(1-x^q qt)^{(\alpha+m+k)/q}} \tag{2.9}$$

and the general operational formula

$$e^{tT_{k,q}}\{x^\alpha f(x)\}=\frac{x^\alpha}{(1-x^q qt)^{(\alpha+k)/q}}f\left[\frac{x}{(1-x^q qt)^{1/q}}\right]. \tag{2.10}$$

We, then, prove that the operator $T_{k,q}$ satisfies the operational relation

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}T_{k,q}^n\{x^{\alpha-nq}f(x)\}=x^\alpha(1+qt)^{\frac{(\alpha+k-q)}{q}}f\{x(1+qt)^{1/q}\}\ . \tag{2.11}$$

Therefore, in view of (2.1)

$${}_\lambda F_\mu\left[\begin{matrix}(\alpha_\lambda);\\(\beta_\mu);\end{matrix}tT_{k,q}\right]x^m=x^m{}_{\lambda+1}F_\mu\left[\begin{matrix}(\alpha_\lambda),\left(\frac{m+k}{q}\right);\\(\beta_\mu);\end{matrix}x^q qt\right] \tag{2.12}$$

and subsequently,

$${}_\lambda F_\mu\left[\begin{matrix}(\alpha_\lambda);\\(\beta_\mu);\end{matrix}tT_{k,q}\right]x^\alpha e^{px^r}=\sum_{j=0}^{\infty}\frac{(p)^j}{j!}x^{\alpha+rj}{}_{\lambda+1}F_\mu\left[\begin{matrix}(\alpha_\lambda),\frac{\alpha+rj+k}{q};\\(\beta_\mu);\end{matrix}x^q qt\right] \tag{2.13}$$

where (α_λ) stands for the sequence of λ parameters $\alpha_1,\alpha_2,\cdots,\alpha_\lambda$; with similar interpretations for (β_μ).

In particular, if we take $(\alpha_\lambda)=(\beta_\mu)-\alpha-k=0$ and $p+1=r-1=q-1=0$, we have

$${}_0F_1\left[-;\alpha+k;tT_k\right]x^\alpha e^{-x}=x^\alpha e^{-x+xt}{}_0F_1\left[-;\alpha+k,-x^2t\right]. \tag{2.14}$$

Defining now $1/T_{k,q}$ as the inverse of the operator $T_{k,q}$ we obtain

$$\frac{1}{T_{k,q}^m}(x^{-\alpha})=\frac{(-1)^m x^{-\alpha-mq}}{q^m\left(\frac{\alpha-k}{q}+1\right)_m} \tag{2.15}$$

and

$$\frac{1}{T_{k,q}}\{(k-q)\log x+1\}=\frac{\log x}{x^q}\ . \tag{2.16}$$

Next, combining (2.1) and (2.15), we deduce

$$\left(\frac{S_{l,q}}{T_{k,q}}\right)^m\left(\frac{y^\beta}{x^\alpha}\right)=\frac{(-1)^m\left(\frac{\beta+l}{q}\right)_m}{\left(\frac{\alpha-k}{q}+1\right)_m}\frac{y^{\beta+mq}}{x^{\alpha+mq}} \tag{2.17}$$

where $S_{l,q}=y^q(l+yD_y)$,

and hence

$$ {}_\lambda F_\mu\left[\begin{matrix}(\alpha_\lambda);\\ (\beta_\mu);\end{matrix}\; t\frac{S_{l,q}}{T_{k,q}}\right]\left(\frac{y^\beta}{x^\alpha}\right)=\frac{y^\beta}{x^\alpha}\,{}_{\lambda+1}F_{\mu+1}\left[\begin{matrix}(\alpha_\lambda),\frac{\beta+l}{q};\\ (\beta_\mu),\frac{\alpha-k}{q}+1;\end{matrix}\; -t(y/x)^q\right]. \tag{2.18}$$

In particular, we have

$$\left(1-t\frac{S_{l,q}}{T_{k,q}}\right)^{-c}\left(\frac{y^\beta}{x^\alpha}\right)=\frac{y^\beta}{x^\alpha}\,{}_2F_1\left[\begin{matrix}c,\frac{\beta+l}{q};\\ \frac{\alpha-k}{q}+1;\end{matrix}\; -t(y/x)^q\right], \tag{2.19}$$

$$\left(1-t\frac{S_{l,q}}{T_{k,q}}\right)^{n}\left(\frac{y^\beta}{x^\alpha}\right)=\frac{y^\beta}{x^\alpha}\,{}_2F_1\left[\begin{matrix}-n,\frac{\beta+l}{q};\\ \frac{\alpha-k}{q}+1;\end{matrix}\; -t(y/x)^q\right], \tag{2.20}$$

$$\left(1-t\frac{S_{l,q}}{T_{k,q}}\right)^{-\left(\frac{\alpha-k}{q}\right)-1}\frac{y^\beta}{x^\alpha}=\frac{y^\beta}{x^\alpha}\{1+t(y/x)^q\}^{-\frac{\beta+l}{q}}, \tag{2.21}$$

and

$$e^{-t/T_{k,q}}(x^{-\alpha})=x^{-\alpha}{}_0F_1\left[-;\frac{\alpha-k}{q}+1;\; t/qx^q\right]. \tag{2.22}$$

3. Operational characterization of the polynomials $M_{\nu n}^{(\alpha)}(x,q)$

Note that the formula (2.2) admits the following equivalent forms

$$(T_{k,q}+x^q\alpha)^n f(x)=x^{nq}\prod_{j=0}^{n-1}(\delta+\alpha+k+jq)f(x), \tag{3.1}$$

and

$$[T_{k,q}+x^q\alpha-x^{q+1}p_\nu'(x)]^n=x^{nq}\prod_{j=0}^{n-1}(\delta+\alpha+k-xp_\nu'(x)+jq). \tag{3.2}$$

If y is a sufficiently differentiable function of x, then (3.2) can also be expressed as

$$x^{-\alpha-nq}e^{p_\nu(x)}T_{k,q}^n\,[x^\alpha e^{-p_\nu(x)}y]=\prod_{j=0}^{n-1}(\delta+\alpha+k-xp_\nu'(x)+jq)y. \tag{3.3}$$

On the other hand,

$$T_q^n\,[x^\alpha e^{-p_\nu(x)}y]=x^{\alpha+nq}e^{-p_\nu(x)}\prod_{j=0}^{n-1}(\delta+\alpha-xp_\nu'(x)+jq)y \tag{3.4}$$

which by an appeal to (2.7) gives

$$T_q^n\,[x^\alpha e^{-p_\nu(x)}y]=\sum_{s=0}^{n}\frac{n!}{s!}x^{\alpha+nq-sq}e^{-p_\nu(x)}M_{\nu(n-s)}^{(\alpha)}(x,q)T_q^s y\ . \tag{3.5}$$

Thus, comparison of (3.4) and (3.5) would yield,

$$\prod_{j=0}^{n-1}(\delta+\alpha-xp_\nu'(x)+jq)y=n!\sum_{s=0}^{n}\frac{x^{-sq}}{s!}M_{\nu(n-s)}^{(\alpha)}(x,q)T_q^s y \tag{3.6}$$

and that of (3.3) and (3.6)

$$\sum_{s=0}^{n}\frac{x^{-sq}}{s!}M_{\nu(n-s)}^{(\alpha+k)}(x,q)\,T_q^s y=\frac{1}{n!}x^{-\alpha-nq}e^{p_\nu(x)}T_{k,q}^n\,[x^\alpha e^{-p_\nu(x)}y]\ . \tag{3.7}$$

In particular, for $y=1$

$$M_{\nu_n}^{(\alpha+k)}(x,q)=\frac{1}{n!}\,x^{-\alpha-nq}\,e^{p_\nu(x)}T_{k,q}^n\,[x^\alpha e^{-p_\nu(x)}]\ . \tag{3.8}$$

In what follows, in view of (2.4) and (2.5), (3.8) can be written as

$$[T_{k,q}+x^q\alpha-x^{q+1}p_\nu'x]^n\cdot 1=n!\,x^{nq}\,M_{\nu n}^{(\alpha+k)}(x,q), \tag{3.9}$$

whereas, by a simple change of variable

$$T_{k,q}^n\,[x^\alpha e^{-p_\nu(\lambda x)}]=x^{\alpha+nq}e^{-p_\nu(\lambda x)}n!\ M_{\nu n}^{(\alpha+k)}(\lambda x,q) \tag{3.10}$$

implying that

$$[T_{k,q}+x^q\alpha-\lambda x^{q+1}p_\nu'(\lambda x)]^n\cdot 1=n!\,x^{nq}\,M_{\nu n}^{(\alpha+k)}(\lambda x,q)\ . \tag{3.11}$$

Next, by operating on both sides of (3.8) with $T_{k,q}^m$ we have

$$T_{k,q}^m\,[x^{\alpha+nq}e^{-p_\nu(x)}\,M_{\nu n}^{(\alpha+k)}(x,q)]=\frac{(m+n)!}{n!}x^{\alpha+(n+m)q}e^{-p_\nu(x)}M_{\nu n}^{(\alpha+k)}(x,q)\ . \tag{3.12}$$

Now, assuming $u=x^\alpha e^{-p_\nu(\lambda x)}$ and $v=x^\beta e^{-p_\nu(\mu x)}$ and applying (2.6), we obtain

$$T_{k,q}^n\,[x\{x^\alpha e^{-p_\nu(\lambda x)}\}\,\{x^\beta e^{-p_\nu(\mu x)}\}]=x\sum_{m=0}^{n}\binom{n}{m}\{T_{k,q}^{n-m}x^\beta e^{-p_\nu(\mu x)}\}\,\{T_{1,q}^m x^\alpha e^{-p_\nu(\lambda x)}\}$$

or, alternatively, the operational formula

$$x^{-\alpha-\beta-1-nq}e^{p_\nu(\lambda x)+p_\nu(\mu x)}\ T_{k,q}^n x^{\alpha+\beta+1}\ e^{-p_\nu(\lambda x)-p_\nu(\mu x)}$$

$$=n!\ \sum_{m=0}^{n}M_{\nu(n-m)}^{(\beta+k)}(\mu x,q)\,M_{\nu m}^{(\alpha+1)}(\lambda x,q)\ . \tag{3.13}$$

Evidently, for $\lambda=\mu=1$, one obtains

$$M_{2\nu n}^{(\alpha+\beta+k+1)}(x,q)=\sum_{m=0}^{n}M_{\nu(n-m)}^{(\beta+k)}(x,q)M_{\nu m}^{(\alpha+1)}(x,q)\ . \tag{3.14}$$

4. Generating functions and recursion formulas

We demonstrate below, how the operator $T_{k,q}$ can be employed to yield, in a very simple manner, a number of generating relations and recursion formulas. For instance, operating by $e^{tT_{k,q}}$ on $x^{\alpha-k}e^{-p_\nu(x)}$, we get

$$e^{tT_{k,q}}[x^{\alpha-k}e^{-p_\nu(x)}]=\sum_{m=0}^{\infty}\frac{t^m}{m!}T_{k,q}^m[x^{\alpha-k}e^{-p_\nu(x)}].$$

Thus, by an appeal to formulas (2.10) and (3.8) and replacing t by $\frac{t}{x^q}$, we are led to the generating relation

$$(1-qt)^{-\alpha/q}e^{p_\nu(x)-p_\nu[x(1-qt)^{-1/q}]}=\sum_{m=0}^{\infty}t^m M_{\nu m}^{(\alpha)}(x,q)\,. \tag{4.1}$$

Again, from (3.8),

$$T_{k,q}^n[x^{\alpha-k}e^{-p_\nu(x)}]=x^{\alpha-k+nq}e^{-p_\nu(x)}n!\;M_{\nu n}^{(\alpha)}(x,q).$$

Operating on both sides by $e^{tT_{k,q}}$ and by a simple change of variables, we get the generating relation

$$\sum_{m=0}^{\infty}\binom{m+n}{n}t^m M_{\nu(m+n)}^{(\alpha)}(x,q)=(1-qt)^{-\alpha/q-n}e^{p_\nu(x)-p_\nu(x(1-qt)^{-1/q})}\times M_{\nu n}^{(\alpha)}\left[\frac{x}{(1-qt)^{1/q}},q\right]. \tag{4.2}$$

This, in a way, generalizes (4.1). Indeed, it reduces to (4.1) when $n=0$.
Next, since

$$\sum_{n=0}^{\infty}t^n M_{\nu n}^{(\alpha-nq)}(x,q)=x^{-\alpha+k}e^{p_\nu(x)}\sum_{n=0}^{\infty}\frac{t^n}{n!}T_{k,q}^n[x^{\alpha-k-nq}e^{-p_\nu(x)}],$$

by virtue of (2.11), we have

$$\sum_{n=0}^{\infty}t^n M_{\nu n}^{(\alpha-nq)}(x,q)=(1+qt)^{\frac{\alpha-q}{q}}e^{p_\nu(x)-p_\nu[x(1+qt)^{1/q}]}\,. \tag{4.3}$$

Multiplying (4.3) by $x^{\alpha-k}e^{-p_\nu(x)}$ and operating on both sides with $T_{k,q}^m$, we obtain

$$T_{k,q}^m\sum_{n=0}^{\infty}t^n x^{\alpha-k}e^{-p_\nu(x)}M_{\nu n}^{(\alpha-nq)}(x,q)=(1+qt)^{\frac{\alpha-q}{q}}T_{k,q}^m[x^{\alpha-k}e^{p_\nu[x(1+qt)^{1/q}]}]$$

so that with the help of (3.12),

$$\sum_{n=0}^{\infty}\binom{m+n}{n}t^n M_{\nu(m+n)}^{(\alpha-nq)}(x,q)=(1+qt)^{\frac{\alpha-q}{q}}e^{p_\nu(x)-p_\nu\{x(1+qt)^{1/q}\}}M_{\nu m}^{(\alpha)}\left[\frac{x}{(1+qt)^{-1/q}},q\right]. \tag{4.4}$$

It is interesting to remark that wheres the formula (4.1) yields

$$M_{\nu n}^{(\alpha)}(x,q)=\sum_{m=0}^{n}\frac{\left(\frac{\alpha-\beta}{q}\right)_m}{m!}q^m M_{\nu(n-m)}^{(\beta)}(x,q), \tag{4.5}$$

a comparison of (4.1) and (4.3) would yield the recursion formula

$$M_{\nu n}^{(\alpha)}(x,-q)=M_{\nu n}^{(\alpha-nq)}(x,q)+q\,M_{\nu(n-1)}^{(\alpha+q-nq)}(x,q). \tag{4.6}$$

Note also that, $T_{k,q}^n\,[x^{\alpha-k}\,e^{-p_\nu(x)}]$ can be expressed as

$T_{k,q}[T_{k,q}^{n-1}\{x^{\alpha-k}e^{-p_\nu(x)}\}]=n!\ x^{\alpha-k+nq}\,e^{-p_\nu(x)}\,M_{\nu n}^{(\alpha)}(x,q)$, and hence

$$n\,M_{\nu n}^{(\alpha)}(x,q)=(xD-xp_\nu'(x)+\alpha-q+nq)\,M_{\nu(n-1)}^{(\alpha)}(x,q) \tag{4.7}$$

a formula, which can be derived from (3.4) as well. On the other hand, writing $T_{k,q}^n\,[x^\alpha e^{-p_\nu(x)}]=T_{k,q}^n\,[x.\,x^m\cdot x^{\alpha-m-1}e^{-p_\nu(x)}]$ and assuming $u=x^{\alpha-m-1}\,e^{-p_\nu(x)}$, $v=x^m$ and making an appeal to (2.6), we get

$$n!\ x^{\alpha+nq}e^{-p(x)}M_{\nu n}^{(\alpha+k)}(x,q)=x\sum_{s=0}^{n}\binom{n}{s}(T_{k,q}^{n-s}x^m)(T_{1,q}^{s}x^{\alpha-m-1}\cdot e^{-p_\nu(x)})\,.$$

This, finally, simplifies to

$$M_{\nu n}^{(\alpha+k)}(x,q)=\sum_{s=0}^{n}\frac{q^s}{s!}\left(\frac{m+k}{q}\right)_s M_{\nu(n-s)}^{(\alpha-m)}(x,q). \tag{4.8}$$

In the particular case when $k=0$, $m=q=1$, it will lead to

$$M_{\nu n}^{(\alpha)}(x,1)=\sum_{s=0}^{n}M_{\nu s}^{(\alpha-1)}(x,1), \tag{4.9}$$

which further reduces to a similar formula for Laguerre polynomials [5, p192] when $\alpha=\alpha+1$ and $p_\nu(x)=x$.

5. Polynomials related to the $M_{\nu n}^{(\alpha)}(x,q)$

In this section, following Carlitz [2,3], we consider a set of polynomials $A_{\nu n}^{(\alpha)}(x,q)$ such that

$$\sum_{m=0}^{n}A_{\nu n}^{(\alpha)}(x,q)\,M_{\nu(n-m)}^{(\alpha+mq)}(x,q)=0,\ n\geq 1,\ \text{and}\ A_0^{(\alpha)}(x,q)=1\,. \tag{5.1}$$

Obviously, the polynomials $A_{\nu n}^{(\alpha)}(x,q)$ are uniquely determined by (5.1). It follows as a consequence of our definition that

$$1=\sum_{n=0}^{\infty}t^n\sum_{m=0}^{n}A_{\nu m}^{(\alpha)}(x,q)\,M_{\nu(n-m)}^{(\alpha+mq)}(x,q)$$
$$=\sum_{m=0}^{\infty}t^m A_{\nu m}^{(\alpha)}(x,q)\sum_{n=0}^{\infty}t^n\,M_{\nu n}^{(\alpha+mq)}(x,q)$$

$$=\sum_{m=0}^{\infty} t^m\ A_{\nu m}^{(\alpha)}(x,q)(1-qt)^{-\frac{(\alpha+mq)}{q}}\ e^{p_\nu(x)-p_\nu[x(1-qt)^{-1/q}]}$$

and hence

$$\sum_{m=0}^{\infty} z^m\ A_{\nu m}^{(\alpha)}(x,q)=(1+qz)^{\frac{\alpha}{q}}\ e^{-p_\nu(x)+p_\nu[x(1+qz)^{1/q}]} \tag{5.2}$$

where $t=z(1-qt)$.

Thus comparison of (4.1) and (5.2) would lead to

$$A_{\nu m}^{(\alpha)}(x,q)=M_{-\nu m}^{(-\alpha)}(x,-q)\ .$$

This indicates that the associated polynomials $A_{\nu n}^{(\alpha)}(x,q)$ have properties similar to those of $M_{\nu n}^{(\alpha)}(x,q)$.

Of particular interest are the Rodrigue's formula

$$A_{\nu n}^{(\alpha)}(x,q)=\frac{1}{n!}\ x^{\alpha+k+nq}\ e^{-p_\nu(x)}\ T_{k,q}^n\ [x^{-\alpha-k}\ e^{p_\nu(x)}] \tag{5.4}$$

which follows from (3.8), and the summation formula

$$M_{\nu n}^{(\alpha)}(x,q)=\sum_{m=0}^{n} M_{2\nu(n-m)}^{(\alpha+\beta)}(x,q)\ A_{\nu m}^{(\beta)}(x,q) \tag{5.5}$$

or equivalently,

$$M_{\nu n}^{(\alpha)}(x,q)=\sum_{m=0}^{n} M_{2\nu(n-m)}^{(\alpha+\beta)}(x,q)\ M_{-\nu m}^{(-\beta)}(x,-q) \tag{5.6}$$

which follows from (4.1).

While concluding we remark that most of our results will correspond to those of Mittal [7] if we set $q=1$ and to those of Al-Salam [1] if we set $k=1$, $q=1$, $p_\nu(x)=x$.

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