Kyungpook Math. J.
Volume 15, Number 2
December, 1975

E-COMPACT CONVERGENCE SPACES AND E-FILTERS

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Engelking and Mrówka have introduced the concept of E-compact spaces for a topological space E(see [2], [9], [10]), i.e. spaces which are homeomorphic to closed subspaces of powers of E. Moreover, for a Hausdorff space E, the full subcategory of E-compact spaces is the epi-reflective hull of E in the category **Haus** of Hausdorff spaces and continuous maps.

In this paper we study E-compact convergence spaces. The natural setting for this is the category **HCon** of Hausdorff convergence spaces and continuous maps. First we identify the epimorphisms of **HCon** (they are not just the dense maps as one might expect at first glance) and the extremal monomorphisms and observe that **HCon** is an (epi, extremal mono) category. Thus, in categorical fashion, several known properties of E-compact topological spaces can be extended at once to convergence spaces e.g. results about epi-reflectiveness.

We introduce the concept of E-filter and use it to give a characterization of E-compact convergence spaces which is new also for the topological case.

Moreover, in the topological case E-filters can be used to construct the E-compactifications.

When E is a regular Hausdorff topological space the E-extendability of a dense morphism in **Haus** is characterized in terms of E-filters.

All concepts of convergence spaces will be used in the sense of Kent and Richardson [8], i.e. X is a convergence space if a relation $x\in\lim\mathscr{F}$ is defined on $X\times$(the set of filters on X) such that

(1) $x\in\lim\dot{x}$ where $\dot{x}$ is the ultrafilter generated by $\{x\}$.

(2) $x\in\lim\mathscr{F}$ and $\mathscr{F}\subset\mathscr{G}$ implies $x\in\lim\mathscr{G}$

(3) $x\in\lim\mathscr{F}$ implies $x\in\lim\mathscr{F}\cap\dot{x}$.

A convergence space is Hausdorff when $\lim\mathscr{F}$ contains at most one point. For categorical background we refer to Herrlich and Strecker [6].

1. Some Properties of HCon

It is well known that the category **HCon** is complete and well-powered and that every such category is an (epi, extremal mono) category (see Theorem 34.5 of

[6]).

DEFINITION 1.1. A map $f:X \to Y$ in **HCon** will be called *t-dense* when there is no proper closed subset of Y containing $f(X)$.

By similar arguments to those in Chapter 15 of [5], we have the following:

PROPOSITION 1.2. *In HCon*
{closed embeddings} = {regular monomorphisms} = {extremal monomorphisms} and {t-dense maps} = {epimorphisms}.

DEFINITION 1.3. Let E be a Hausdorff convergence space. A convergence space is said to be *E-compact* if it is isomorphic to a closed subspace of a power of E.

For any $E \in$ **HCon**, every epi-reflective subcategory of **HCon** contains all E-compact spaces whenever it contains E, for **HCon** is complete. The following result generalizes the known corresponding fact for Hausdorff topological spaces.

THEOREM 1.4. *For any $E \in$ Hcon, the full subcategory of E-compact spaces is epi-reflective in the category HCon.*

PROOF. Let G: **HCon**$(E) \to$ **HCon** be the embedding functor of the full subcategory of E-compact spaces. A routine verification shows that G preserves limits. Thus by the special adjoint functor theorem (28.11, [6]) G has a left adjoint i.e. **HCon**(E) is reflective. On forming the (epi, extremal mono)-factorization of the reflection map of each object of **HCon**, it becomes clear that **HCon**(E) is epi-reflective in **HCon**.

REMARK. Herrlich has generalized the concept of E-compact spaces to that of $\mathscr{E}$-compact spaces for a class $\mathscr{E}$ of Hausdorff topological spaces. He has shown that the full subcategory of $\mathscr{E}$-compact spaces is the epireflective hull of $\mathscr{E}$ in the category **Haus** ([4]). The corresponding conclusion cannot be made for **HCon** since this category is not co-(well-powered) (see [11]).

2. E-compactness and E-filters

Throughout this section X and E will be Hausdorff convergence spaces and $C(X, E)$ will denote the set of continuous maps on X into E. $\underline{I}$ will denote the closed unit interval of the real line R.

It is known [1] that a completely regular space X is compact ($=\underline{I}$-compact) iff every maximal completely regular filter on X is convergent and that every maximal completely regular filter has a convergent image under any f in $C(X, \underline{I})$.

DEFINITION 2.1. A filter $\mathscr{F}$ on X is an *E-filter* if $f(\mathscr{F})$ is convergent for any f in $C(X, E)$.

Obviously convergent filters are E-filters and for a compact Hausdorff space E, all ultrafilters are E-filters.

LEMMA 2.2. *Suppose $C(X, E)$ separates the points of X and every E-filter on X is convergent. Then X has the initial convergence structure determined by $C(X, E)$.*

PROOF. Let g be a map on a convergence space Y into X such that fg is continuous for all $f \in C(X, E)$. Let $\mathscr{F}$ be a filter on X which converges to x. Then for any $f \in C(X, E)$, $fg(\mathscr{F})$ converges to $fg(x)$. Hence $g(\mathscr{F})$ is an E-filter and is convergent by the assumption. Since $C(X, E)$ separates the points of X, $g(\mathscr{F})$ converges to $g(x)$. Thus g is continuous as required.

THEOREM 2.3. *X is E-compact iff $C(X, E)$ separates the points of X and every E-filter on X is convergent.*

PROOF. Let $\mathfrak{A}$ be the class of convergence spaces which satisfy the conditions in the theorem. We claim that $\mathfrak{A}$ is productive and closed hereditary. To see this, let F be a closed subspace of a member X of $\mathfrak{A}$. Clearly $C(F, E)$ separates the points of F. Let $\mathscr{F}$ be an E-filter on F. Since a continuous image of an E-filter is again an E-filter, the filter $\overline{\mathscr{F}}$ generated by $\mathscr{F}$ on X is an E-filter. Let x be the limit of $\overline{\mathscr{F}}$. Since F is closed, x belongs to F and hence $\mathscr{F}$ is convergent.

Let $(X_i)_{i \in I}$ be a family in $\mathfrak{A}$ and $(\prod X_i, p_i)$ its product. Obviously $C(\prod X_i, E)$ separates the points of $\prod X_i$. Let $\mathscr{F}$ be an E-filter on $\prod X_i$. For each $i \in I$, $p_i(\mathscr{F})$ is a convergent E-filter on X_i. Hence $\mathscr{F}$ is convergent. Since E belongs to $\mathfrak{A}$, every E-compact space belongs to $\mathfrak{A}$.

Conversely, let X be a member of $\mathfrak{A}$ and e: $X \to E^{C(X,E)}$ the parametric map into the product space $E^{C(X,E)}$. Since $E^{C(X,E)}$ has the initial structure determined by the projection maps p_f, $(f \in C(X, E))$ the above lemma leads to the conclusion that X has the initial structure determined by e i.e. e is an embedding. To show that e is a closed embedding, let z be in the closure of $e(X)$. Then there is a filter $\mathscr{F}$ on $E^{C(X,E)}$ such that $\mathscr{F}$ converges to z and $e(X) \in \mathscr{F}$. Hence $e^{-1}(\mathscr{F})$ is a filter base on X. For each $f \in C(X, E)$, $f(e^{-1}(\mathscr{F}))=p_f e(e^{-1}(\mathscr{F}))$ contains $p_f(\mathscr{F})$ so that $f(e^{-1}(\mathscr{F}))$ converges to $p_f(z)$. Hence $e^{-1}(\mathscr{F})$ is an E-filter and $e^{-1}(\mathscr{F})$ is convergent, say to $x \in X$. Since $p_f e(x)=p_f e(\lim e^{-1}(\mathscr{F}))$

$=\lim p_f e(e^{-1}(\mathscr{F}))=p_f(z)$, $e(x)=z$ belongs to $e(X)$ as required.

3. E-compact topological spaces

Henceforth E will be a Hausdorff topological space. In this special case the preceding results can be somewhat extended and related to known special results.

LEMMA 3.1 *Let E be a Hausdorff topological space. A filter $\mathscr{F}$ on a convergence space X is an E-filter iff $\mathscr{F}$ contains an open E-filter.*

PROOF. Given the E-filter $\mathscr{F}$, put $x_f=\lim f(\mathscr{F})$ $(f\in C(X,E))$ and note that the open neighborhood filter $O(x_f)$ is contained in $f(\mathscr{F})$. Let $\mathscr{G}$ be the filter generated by the join of the open filter bases $f^{-1}O(x_f)$. Then $\mathscr{G}$ is an open E-filter contained in $\mathscr{F}$.

$\underline{I}$-filters and R-filters can be characterized in terms of *completely regular* filters $\mathscr{F}$ on a convergence space X (i.e. $\mathscr{F}$ which have an open base $\mathscr{B}$ such that for each $B\in\mathscr{B}$ there exists $C\in\mathscr{B}$ and $f\in C(X,\underline{I})$ such that $C\subset B$, f is 0 on C and 1 on $X-B$).

PROPOSITION 3.2. *Let $\mathscr{F}$ be a filter on a convergence space X. Then*

(1) *$\mathscr{F}$ is an $\underline{I}$-filter iff $\mathscr{F}$ contains a maximal completely regular filter.*

(2) *$\mathscr{F}$ is an R-filter iff $\mathscr{F}$ contains a maximal completely regular filter with the countable intersection property.*

PROOF. (1) The open $\underline{I}$-filter $\mathscr{G}$ constructed in the proof of 3.1 is a completely regular filter contained in $\mathscr{F}$. Moreover $\mathscr{G}$ is a maximal such filter. Indeed, suppose there is a completely regular filter $\mathscr{G}'$ which contains $\mathscr{G}$ properly. Let G be an open set belonging to $\mathscr{G}'-\mathscr{G}$. Then there is an open set $F\in\mathscr{G}'$ with $F\subset G$ and a continuous map $f:X\to\underline{I}$ with $f(F)=0$ and $f(X-G)=1$. It is obvious that $f(\mathscr{G}')$ converges to 0. Since every member of $\mathscr{G}$ meets $f^{-1}((z,1])$ for each $z<1$ and $f(\mathscr{G})$ is convergent, $f(\mathscr{G})$ converges to 1. Hence $f(\mathscr{G}')$ also converges to 1, which is a contradiction. For the converse, it is easy to show that every maximal completely regular filter is an $\underline{I}$-filter (also see [1]) and hence a filter containing such a filter is again an $\underline{I}$-filter.

(2) Using the fact that every filter on R with the countable intersection property has a cluster point, one can easily show that every maximal completely regular filter with the countable intersection property is an R-filter.

Conversely, let $\mathscr{F}$ be an R-filter on a space X. Let $c:X\to cX$ be the complete regularization of X. Then c is onto. Since $c(\mathscr{F})$ is again an R-filter on cX, it is a Cauchy filter on the uniform space cX with the initial uniform structure determined by $C(cX, R)$. Hence $c(\mathscr{F})$ contains a maximal completely regular filter $\mathscr{G}$ with the countable intersection property because the minimal Cauchy filters on the space cX are exactly maximal completely regular filters with the countable intersection property ([7]). It is easy to show that $c^{-1}(\mathscr{G})$ is a maximal completely regular filter with the countable intersection property and $\mathscr{F}$ contains $c^{-1}(\mathscr{G})$.

In the special case $E=\underline{I}$ 2.3 and 3.2 give the following strengthening of the known characterization of compactness in terms of completely regular filters (see [1]). A convergence space X is compact Hausdorff topological iff $C(X, \underline{I})$ separates points and every maximal completely regular filter on X is convergent. Obviously a similar characterization of realcompact X can be concluded from 2.3 and 3.2.

DEFINITION 3.3. Let $\mathscr{F}$ be an E-filter on a convergence space X. We define a map $[\mathscr{F}]: C(X, E) \to E$ by $[\mathscr{F}](f) = \lim f(\mathscr{F})$. Then $[\mathscr{F}]$ will be called an *E-filter map* on X.

Let rX be the set of all E-filter maps on X and $r: X \to rX$ the map defined by $r(x) = [\dot{x}]$.

For any $f \in C(X, E)$, there is a map s_f: $rX \to E$ defined by $s_f([\mathscr{F}]) = \lim f(\mathscr{F})$.

The convergence space with the initial convergence structure on rX determined by $(s_f)_{f \in C(X, E)}$ will be denoted again by rX. Since $s_f r(x) = s_f([\dot{x}]) = \lim f(\dot{x}) = f(x)$ for each $x \in X$, we have $s_f r = f$ for each $f \in C(X, E)$. Hence r is continuous. Considering the map $e: rX \to E^{C(X, E)}$ defined by $e([\mathscr{F}]) = (\lim f(\mathscr{F}))_{f \in C(X, E)}$, one can easily prove that *e is a closed embedding and that $r: X \to rX$ is precisely the **Haus**(E)-reflection of X*, where **Haus**(E) denotes the full subcategory of **Haus** determined by all E-compact spaces.

REMARK. For a Hausdorff convergence space E, one might consider the space rX of E-filter maps. However, rX is isomorphic to the closure of $c(X)$, where $c: X \to E^{C(X, E)}$ denotes the parametric map defined by $p_f c = f$ for each $f \in C(X, E)$ and projection p_f. Since the closure of a subset in a convergence space need not be closed, rX need not be E-compact.

DEFINITION 3.4. A map $f: X \to Y$ is said to be *E-extendable* if for any $h \in$

$C(X,E)$ there is a map $g\in C(Y,E)$ with $gf=h$.

It is known [10] that an E-regular space X, i.e. a space which is homeomorphic with a subspace of a power of E, is E-compact iff X has no proper dense E-extendable extension. Such extensions can also be characterized in terms of E-filters.

THEOREM 3.5. *Let E be a regular Hausdorff space and let $e:X\to Y$ be a continuous map between topological spaces. Then the following are equivalent:*

(a) e is dense and E-extendable

(b) for any open E-filter $\mathscr{G}$ on Y, $e^{-1}(\mathscr{G})$ is an open E-filter base on X

(c) for any E-filter $\mathscr{G}$ on Y, the non-empty members of $e^{-1}(\mathscr{G})$ form an E-filter base on X.

PROOF. In view of 3.1 it is enough to show the equivalence of (a) and (b). Suppose e is dense and E-extendable. For any open E-filter $\mathscr{F}$ on Y, $e^{-1}(\mathscr{F})$ is a filter base. For any $f\in C(X,E)$, there is a map $\bar{f}\in C(Y,E)$ with $\bar{f}e=f$. Since $f(e^{-1}(\mathscr{F}))$ contains $\bar{f}(\mathscr{F})$, $f(e^{-1}(\mathscr{F}))$ is convergent.

Conversely, for any $y\in Y$, the open neighborhood filter $O(y)$ of y is an open E-filter. Hence $e^{-1}(O(y))$ is again an open E-filter so that e is dense. For any $f\in C(X,E)$ and $y\in Y$, define $\bar{f}(y)=\lim f(e^{-1}(O(y)))$. Since $\bar{f}e(x)=\lim f(e^{-1}(O(e(x))))=\lim f(O(x))=f(x)$, we have $\bar{f}e=f$. For any $y\in Y$, let U be a closed neighborhood of $\bar{f}(y)$ in E. Then there is an open neighborhood V of y with $f(e^{-1}(V))\subset U$. Since for any $z\in V$, $\bar{f}(z)=\lim(\{f(e^{-1}(V\cap W))\mid W\in O(z)\})$, $\bar{f}(z)\in f(e^{-1}(V))\subset U$, i.e. $\bar{f}(V)\subset U$. Hence $\bar{f}$ is continuous.

COROLLARY 3.6. A dense continuous map $f\colon X\to Y$ is $\underline{I}$-extendable (resp. R-extendable) iff the inverse image of every maximal completely regular filter (with the countable intersection property) under f is again such a filter on X.

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Financial assistance from the National Research Council of Canada is gratefully acknowledged.

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