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ON AN INTEGRAL TRANSFORM INVOLVING A KERNEL OF MELLIN-BARNES TYPE INTEGRAL

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1. Introduction

Recently Kalla [3] has considered an integral transform over the interval $(0, \infty)$ as,

$$g(x)=\int_0^\infty S_{p,q,r}(xu)h(u)du \tag{1}$$

where

$$S_{p,q,r}(x)=\frac{1}{2\pi i}\int_C P(s)x^{-s}ds \tag{2}$$

and

$$P(s)=\frac{\prod_{j=1}^{p}\Gamma\left(\frac{a_j+As}{p}\right)\prod_{j=1}^{q}\Gamma\left(\frac{b_j+B_js}{m_j}\right)\prod_{j=1}^{r}\Gamma\left(\frac{1+d_j-D_js}{n_j}\right)}{\prod_{j=1}^{q}\Gamma\left(\frac{c_j+B_js}{m_j}\right)\prod_{j=1}^{r}\Gamma\left(\frac{1+c_j-D_js}{n_j}\right)}. \tag{3}$$

The following are the conditions of validity of (1):

(i) $h(x)\in L_2(0,\infty)$;

(ii) $x>0$;

(iii) p and r are positive integers and q is a non negative integer;

(iv) $m_j>0$ for $j=1,\cdots,q$

(v) $n_j>0$ for $j=1,\cdots,r$

(vi) the contour C is a straight line parallel to the imaginary axis in the complex s-plane given by $s=\frac{1}{2}+it$, t being real and $-\infty<t<\infty$ and all the poles of $\Gamma\left(\frac{a_j+As}{p}\right)$ for $j=1,\cdots,p$ and $\Gamma\left(\frac{b_j+B_js}{m_j}\right)$ for $j=1,\cdots,q$ must lie to the left of C while those of $\Gamma\left(\frac{1+d_j-D_js}{n_j}\right)$ for $j=1,\cdots,r$ to the right of it;

(vii) $a_i\neq a_j$, $i\neq j$, $i=1,\cdots,p$. Similar conditions hold for all b_j and c_j, $j=1,\cdots,q$ and d_j and c_j, $j=1,\cdots,r$.

(viii) $|\arg x| < \frac{1}{2}\frac{A}{p}\pi$

$h(t)=o(t^{\xi})$ for small t, and $h(t)=o(t^{\eta}.e^{-\beta t})$ for large t.

$$\mathrm{R}\left(\xi+\frac{a_j}{A}+\frac{b_i}{B_i}+1\right)>0,\ j=1,\cdots,p;\ i=1,\cdots,q,\ \mathrm{R}(\beta)>0.$$

He has represented the function $S_{p,q,r}$ as:

$$S_{p,q,r}\left(x\left|\begin{matrix}(c_1,B_1),\cdots,(c_q,B_q);\ (e_1,D_1),\cdots,(e_r,D_r)\\(a_1,A),\cdots,(a_p,A);\ (b_1,B_1),\cdots,(b_q,B_q);\ (d_1,D_1),\cdots,(d_r,D_r)\end{matrix}\right.\right). \quad (4)$$

In what follows for the sake of brevity (a_p, A_p) represent the set of parameters (a_1, A_1), $,(a_p, A_p)$.

We shall denote symbolically (1) as $S[h(u)]=g(x)$.

The object of the present paper is two fold, firstly to consider some simple properties of the function $S_{p,q,r}(x)$ and secondly to establish certain theorems involving the transform (1).

2. Some properties of $S_{p,q,r}(x)$

(i) $S_{p,q,r}(x)=o\left[x^{\min\left(\frac{a_j}{A}+\frac{b_i}{B_i}\right)}\right]$ for small x, $j=1,\cdots,p$; $i=1,\cdots,q$ and

$S_{p,q,r}(x)=o\left(x^{-\frac{n_j+d_j}{D_j}}\right)$ for large x, $j=1,\cdots,r$.

(ii) The function $S_{p,q,r}(x)$ is symmetric in the parameters $b_1,\cdots,b_q$; $c_1,\cdots,c_q$; $d_1,\cdots,d_r$; and $e_1,\cdots,e_r$. Thus, if one of the b_j's, $j=1,\cdots,q$ is equal to one of the c_i's, $i=1,\cdots,q$, the function reduces to one of lower order.

For example,

$$S_{p,q,r}\left(x\left|\begin{matrix}(c_1,B_1),\cdots,(c_q,B_q);\ (e_1,D_1),\cdots,(e_r,D_r)\\(a_1,A),\cdots,(a_p,A);\ (c_1,B_1),(c_2,B_2),\cdots,(b_q,B_q);\ (d_1,D_1),\cdots,(d_r,D_r)\end{matrix}\right.\right)$$
$$=S_{p,q-1,r}\left(x\left|\begin{matrix}(c_2,B_2),\cdots,(c_q,B_q);\ (e_1,D_1),\cdots,(e,D_r)\\(a_1,A),\cdots,(a_p,A);\ (b_2,B_2),\cdots,(b_q,B_q);\ (d_1,D_1),\cdots,(d_r,D_r)\end{matrix}\right.\right). \quad (5)$$

Similarly, if one of the d_j's, $j=1,\cdots,r$ is equal to one of the e_i's, $i=1,\cdots,r$, the function reduces to one of lower order, to obtain $S_{p,q,r-1}(x)$.

(iii) The following identities can be established easily from the definition:

$$S_{p,q,r}\left(\frac{1}{x}\right)=H^{r,p+q}_{r+p+q,q+r}\left[x\left|\begin{matrix}\left(1-\frac{a_1}{p},\frac{A}{p}\right),\cdots,\left(1-\frac{a_p}{p},\frac{A}{p}\right),\left(1-\frac{b_1}{m_1},\frac{B_1}{m_1}\right),\cdots,\\\left(\frac{1+d_1}{n_1},\frac{D_1}{n_1}\right),\cdots,\left(\frac{1+d_r}{n_r},\frac{D_r}{n_r}\right);\ \left(1-\frac{c_1}{m_1},\frac{B_1}{m_1}\right),\cdots,\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\frac{b_q}{m_q}, \frac{B_q}{m_q}\right); \left(\frac{1+e_1}{n_1}, \frac{D_1}{n_1}\right), \cdots, \left(\frac{1+e_r}{n_r}, \frac{D_r}{n_r}\right) \\ \left(1-\frac{c_q}{m_q}, \frac{B_q}{m_q}\right)\end{matrix}\right] \tag{6}$$

where on the right appears Fox's H-function [2].

$$\begin{aligned} &x^{\sigma}\cdot S_{p,q,r}(x) \\ &=S_{p,q,r}\left(x \left| \begin{matrix}(c_1+\sigma B_1, B_1), \cdots, (c_q+\sigma B_q, B_q);\ (e_1-\sigma D_1, D_1), \cdots, \\ (a_1+\sigma A, A), \cdots, (a_p+\sigma A, A);\ (b_1+\sigma B_1, B_1), \cdots, (b_q+\sigma B_q, B_q);\end{matrix}\right.\right. \\ &\qquad \left.\begin{matrix}(e_r-\sigma D_r, D_r) \\ (d_1-\sigma D_1, D_1),\ ,(d-\sigma D_r, D_r)\end{matrix}\right) \end{aligned} \tag{7}$$

and for $\sigma>0$:

$$\begin{aligned} &S_{p,q,r}\left(x^{\sigma} \left| \begin{matrix}(c_1, \sigma B_1), \cdots, (c_q, \sigma B_q);\ (e_1, \sigma D_1), \cdots, (e_r, \sigma D_r) \\ (a_1, \sigma A), \cdots, (a_p, \sigma A),\ (b_1, \sigma B_1), \cdots, (b_q, \sigma B_q);\ (d_1, \sigma D_1), \cdots, (d_r, \sigma D_r)\end{matrix}\right.\right) \\ &=S_{p,q,r}(x) \end{aligned} \tag{8}$$

(iv) The Laplace transform of the function $S_{p,q,r}$ (x). shall be denoted symbolically as $L\{f(t);\ p\}$ the classical Laplace transform:

$$\phi(p)=p\int_0^{\infty} e^{-pt}h(t)\ dt \tag{9}$$

THEOREM.

$$L\{S_{p,q,r}(t);\ p\}=S_{p,q,r+1}\left(\frac{1}{p} \left| \begin{matrix}\{(c_q, B_q)\};\ \{(e_r, D_r)\} \\ \{(a_p, A)\};\ \{(b_q, B_q)\};\ \{(d_r, D_r)\};\ (0,1)\end{matrix}\right.\right) \tag{10}$$

where $\mathrm{R}\left(\frac{a_j}{A}+\frac{b_i}{B_i}+1\right)>0,\ j=1, \cdots, p;\ i=1, \cdots, q;\ \mathrm{R}(p)>0$.

PROOF. By (9) and (2) we have

$$\begin{aligned} \phi(p)&=p\int_0^{\infty} e^{-pt}\ S_{p,q,r}(t)\ dt \\ &=p\int_0^{\infty} e^{-pt}\left[\frac{1}{2\pi i}\int_C P(s)t^{-1}ds\right]dt \end{aligned}$$

then changing the order of integration and a little simplification, we obtain the result (10).

3. Theorems

In this section we shall establish certain on S-transform (1).

THEOREM 1.
If

$$S[f(u)]=g(x)$$

then

$$S[f(ax)]=\frac{1}{a}g\left(\frac{x}{a}\right) \tag{11}$$

provided that the S-transform of $|f(u)|$ *exist and* $a>0$.

The proof of the above theorem is obvious.

THEOREM 2.
If

$$S[h(u)]=\varphi(x)$$

and

$$S[f(u)]=\phi(x)$$

then

$$\int_0^\infty \phi(u)f(u)du=\int_0^\infty h(u)\phi(u)du \tag{12}$$

provided that the S-transform of $|h(u)|$ *and* $|f(u)|$ *exist and either of the integrals* (12) *is absolutely convergent.*

PROOF. We have

$$\int_0^\infty \phi(x)f(x)dx=\int_0^\infty f(x)\left[\int_0^\infty S_{p,q,r}(xu)h(u)du\right]dx$$

then changing the order of integration, which is permissible due to the absolute convergence of the integrals involved [1], we obtain the desired result.

The following theorem establish a relation between the integral transform (1) and the Varma transform [5]:

$$V_{k,m}\{f(t);p\}=p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{m-\frac{1}{2}}W_{k,m}(pt)f(t)dt.$$

THEOREM 3.
If

$$\phi(x)=\int_0^\infty S_{p,q,r}(xu)h(u)du$$

then

$$V_{k,m}\{g(x)\phi(x);p\}=\int_0^\infty h(u)V_{k,m}\{g(x)\ S_{p,q,r}(xu);p\}\,du \tag{13}$$

provided that the S-transform of $|h(u)|$ *exist,* $g(x)=0(x^{\xi'})$ *for small* x *and* $g(x)=o(x^{\eta'}e^{-\beta' x})$ *for large* x,

$$\mathrm{R}\left\{m\pm m+\xi'+\frac{a_j}{A}+\frac{b_i}{B_i}+1\right\}>0,\ j=1,\ ,p;\ i=1,\cdots,q;\mathrm{R}(p+\beta')>0.$$

PROOF. We have

$$V_{k,m}\{g(x)\phi(x);p\}$$

$$=p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)\Big[g(x)\int_0^\infty S_{p,q,r}(xu)h(u)du\Big]dx$$

$$=\int_0^\infty h(u)\Big[p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)g(x)\,S_{p,q,r}(xu)dx\Big]du.$$

The change of order of integration is justified by de la Vallée Poussin's theorem [1] under the conditions stated with the theorem.

Relationship with the Mellin transform.
We prove a relationship between the Mellin transform of $f(x)$ defined by [4]

$$\bar{f}(s)=\int_0^\infty x^{s-1}\ f(x)dx \tag{14}$$

and the Mellin transform of its image $\phi[f;p]$ in the S-transform (1), that is

$$\bar{\phi}(s)=\int_0^\infty p^{s-1}\ \phi[f;p]\,dp. \tag{15}$$

If

$$\phi(x)=\int_0^\infty S_{p,q,r}(xu)h(u)du$$

by (15) we have

$$\bar{\phi}(s)=\int_0^\infty p^{s-1}\phi(p)dp$$

$$=\int_0^\infty p^{s-1}\Big\{\int_0^\infty S_{p,q,r}(px)h(x)dx\Big\}dp$$

$$=\int_0^\infty h(x)x^{-s}dx\int_0^\infty u^{s-1}\, S_{p,q,r}(u)du.$$

on changing the order of integration and a little simplification. Since the x-integral and u-integral above are independent of each other, we are lead to the following theorem:

THEOREM 4. *If the Mellin transform of* $|h(x)|$ *and* $|\phi(p)|$ *exist then*

$$\bar{\phi}(s)=\bar{f}(1-s)P(s) \tag{16}$$

where $P(s)$ *is defined by* (3), *and provided that the S-transform of* $h(x)$ *exist,* $\mathrm{R}\left(s+\frac{a_j}{A}+\frac{b_i}{B_i}\right)>0,\ j=1,\cdots,p;\ i=1,\cdots,q.$

The following theorem establishes a connection between Laplace and S-transform.

THEOREM 5. *If S-transform of* $|h(u)|$ *exist, and* $\mathrm{R}\left\{\xi+\frac{a_j}{A}+\frac{b_i}{B_i}+1\right\}>0,\ j=1,$ $\cdots,p;\ i=1,\cdots,q;\ \mathrm{R}(\beta+p)>0$ *then*

$$L\{\phi(t);p\}=\int_0^\infty h(u)S_{p,q,r}\left(\frac{u}{p}\,\middle|\,\begin{matrix}\{(c_q,B_q)\};\ \{(e_r,D_r)\}\\ \{(a_p,A)\};\ \{(b_q,B_q)\};\ \{(d_r,B_r)\};\ (0,1)\end{matrix}\right)du. \tag{17}$$

PROOF. By (1) and (9) we have

$$L\{\phi(t);p\}=p\int_0^\infty e^{-pt}\left[\int_0^\infty S_{p,q,r}(tu)h(u)du\right]dt$$

$$=p\int_0^\infty h(u)du\int_0^\infty e^{-pt}\, S_{p,q,r}(tu)dt$$

with the help of the result (10) we obtain (17).

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