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TOPOLOGY OF FRAMED MANIFOLDS

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0. Introduction

D.E. Blair [1], S.I. Goldberg and K. Yano [4] have studied the framed manifolds with f-structure. This is a generlization of almost complex manifold and almost contact manifold. In a framed manifold, we take an interest in S-structure the analogue of Kaehler structure in almost complex manifolds and of the Sasakian structure in almost contact manifolds.

In this paper we shall discuss harmonic 1-forms in compact framed S-manifold and obtain some analogous results to compact Kaehlerian manifold and compact Sasakian manifold. The main theorems of the paper are Theorems 3.3, 4.3, 5.1 and 6.5. In §1 we give definitions of framed manifolds. In §2 for later use we give preliminary formulas on framed S-manifold and framed C-manifold. In §3 we discuss harmonic 1-form and we shall prove Theorem 3.3. In §4 we have used $2n$-homothetic deformations to get the results on first Betti number and we shall prove Theorem 4.3. In §5 we discuss the relations of harmonic 1-forms and the sectional curvatures and prove Theorem 5.1. In §6 we consider an f-holomophic pinching to get the results on first Betti number and we prove Theorem 6.5.

1. Framed manifolds

Let M be a $(2n+s)$-dimensional differentiable manifolds with an f-structure of rank $2n$. If there exist on M vector fields ξ_α and 1-forms η_α such that

(1.1) $$f^2=-I+\Sigma\,\xi_\alpha\otimes\eta_\alpha,$$

(1.2) $$\eta_\alpha(\xi_\beta)=\delta_{\alpha\beta},$$

(1.3) $$f\xi_\alpha=0,\quad \eta_\alpha\circ f=0\,,$$

where the indices α,β run over the range $\{1,2,\cdots,s\}$ and repeated index α is to be summed from 1 to s, then we call the structure a framed structure and the manifold M is called a globally framed f-manifold or a framed manifold ([1], [3], [4]).

The framed manifold M is called a framed metric manifold if there exists on M a Riemannian metric g such that

(1.4) $$g(fX, fY)=g(X, Y)-\Sigma\ \eta_\alpha(X)\eta_\alpha(Y),$$

for vector fields X and Y on M, where we put $\eta_\alpha(X)=g(\xi_\alpha, X)$.

If the tensor field S of type (1, 2) defined by

$$S(f)=[f,f]+\Sigma\ \xi_\alpha\otimes d\eta_\alpha$$

vanishes identically, the framed structure is said to be normal and the manifold M is called a normal framed manifold.

Further a framed metric structure which is normal and has closed fundamental 2-form F, that is,

(1.5) $$dF=0, \quad F(X,Y)=g(X,fY),$$

will be called a framed K-structure and M a framed K-manifold. It should be noted that a framed K-manifold is orientable since

$$\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_s\wedge F^n\neq 0\,.$$

There are special two types of framed K-manifold [1]:

1) If there exists global linearly independent 1-forms $\eta_1, \cdots, \eta_s$ such that $d\eta_1=\cdots=d\eta_s=2F$, then we call the structure a framed S-structure and the manifold M a framed S-manifold. As example, there is Sasakian structure for $s=1$.

2) If there exists global 1-forms $\eta_1, \cdots, \eta_s$ on a framed K-manifold M such that $d\eta_1=\cdots=d\eta_s=0$, then we call the structure a framed C-structure and M a framed C-manifold. As example, there is cosymplectic structure for $s=1$.

2. Identities in framed S-manifold

In this section, we prepare identities in a $(2n+s)$-dimensional framed S-manifold for later use.

We denote by L the operator of Lie derivative, then the following properties are well-known [1], [3].

(2.1) $$L(\xi_\alpha)g=0\,,$$

(2.2) $$L(\xi_\alpha)f=0,\ L(\xi_\alpha)F=0\,.$$

From (2.1) we see that the vector fields $\xi_1, \cdots, \xi_s$ are Killing.

Denoting covariant differentiation by ∇ in a framed K-manifold we get $(d\eta_\alpha)_{ji}=\nabla_j\eta_{\alpha i}$. Thus, on a framed S-manifold we have

(2.3) $$\nabla_j\eta_{\alpha i}=f_{ji}\,,$$

and in the case of a framed C-manifold

(2.4) $$\nabla_j\eta_{\alpha i}=0\,.$$

Differentiating covariantly f_{ji} on a framed S-manifold, by a longthy computation we have [1]

(2.5) $$\nabla_k f_{ji}=\Sigma_\alpha(\eta_{\alpha j}g_{ik}-\eta_{\alpha i}g_{jk})-\Sigma_{\alpha,\beta}\,\eta_{\beta k}(\eta_{\alpha j}\eta_{\beta i}-\eta_{\alpha i}\eta_{\beta j}),$$

From (2.5) we have

(2.6) $$\nabla^\alpha f_{\alpha i}=2n\Sigma_\alpha\eta_{\alpha i}, \qquad \nabla^a\nabla_a f_{ji}=-2sf_{ji},$$

in the case of a framed C-manifold

(2.7) $$\nabla_k f_{ji}=0 .$$

Next, applying the Ricci's identity to $\eta_{\alpha i}$, we get

$$\nabla_k\nabla_j\eta_{\alpha i}-\nabla_j\nabla_k\eta_{\alpha i}=-R_{kji}{}^t\eta_{\alpha t} .$$

Substituting (2.3) and (2.5) into the last equation, we have

(2.8) $$R_{kji}{}^t\eta_{\alpha t}=\Sigma_\alpha(\eta_{\alpha k}g_{ji}-\eta_{\alpha j}g_{ik})-\Sigma_{\alpha,\beta}\,\eta_{\beta i}(\eta_{\alpha j}\eta_{\beta k}-\eta_{\alpha k}\eta_{\beta j}) .$$

Transvecting g^{ji} to (2.8) we have

(2.9) $$R_k{}^t\eta_{\alpha t}=2n\Sigma_\alpha\eta_{\alpha k} .$$

Similarly, applying the Ricci's identity to $f_i{}^h$, we get

$$\nabla_k\nabla_j f_i{}^h-\nabla_j\nabla_k f_i{}^h=R_{kjt}{}^h f_i{}^t-R_{kji}{}^t f_t{}^h.$$

Substituting (2.5) into the last equation, we have

$$\begin{aligned}R_{ihkj}f_i{}^h=&-R_{kjit}f_h{}^t+s(f_{ki}g_{jh}+f_{jh}g_{ki}-f_{kh}g_{ji}-f_{ji}g_{kh})\\&+2\Sigma_{\alpha,\beta}f_{kj}(\eta_{\alpha h}\eta_{\beta i}-\eta_{\alpha i}\eta_{\beta h})\\&-s\Sigma_\beta(f_{ki}\eta_{\beta j}\eta_{\beta h}-f_{kh}\eta_{\beta j}\eta_{\beta i}-f_{ji}\eta_{\beta k}\eta_{\beta h}+f_{jh}\eta_{\beta i}\eta_{\beta k})\\&-\Sigma_{\alpha,\beta}(f_{kh}\eta_{\alpha i}\eta_{\beta j}-f_{jh}\eta_{\alpha i}\eta_{\beta k}-f_{ki}\eta_{\alpha h}\eta_{\beta i}+f_{ji}\eta_{\alpha h}\eta_{\beta k})\end{aligned}$$

Transvecting g^{kh} to the last equation we have

(2.10) $$\frac{1}{2}R_{stji}f^{st}=R_{tj}f_i{}^t+(2n-1)sf_{ji} .$$

Since R_{stji} and f_{ji} are skew-symmetric with respect to j and i in (2.10), we have

(2.11) $$R_{jt}f_i{}^t=-R_{it}f_j{}^t .$$

From (2.10) we have

(2.12) $$f^{st}\nabla_s\nabla_t u_j=-\{R_{jt}f_i{}^t+(2n-1)sf_{ji}\}u^i,$$

for any vector u_j .

3. Harmonic 1-forms in a compact framed S-manifold

In this section, we consider 1-form and first Betti number in a compact framed S-manifold.

First we prove

LEMMA 3.1. *In a compact framed S-manifold, a harmonic 1-form w is orthogonal to* ξ_α, *that is,* $\xi^i_\alpha w_i=0$.

PROOF. Since the vector fields ξ_α are Killing, we have $dC_\alpha=0$ for the scalars C_α defined by $C_\alpha=\xi^i_\alpha w_i$ for each α, Hence C_α are constant. If we define u by

$$w=\Sigma C_\alpha \eta_\alpha+u\ , \tag{3.1}$$

then u is a 1-form orthogonal to ξ_α. Operating Δ to the last equation we get $\Delta u=-\Sigma C_\alpha \Delta\eta_\alpha$ and as η_α are Killing we have

$$(\Delta u)_i=-\Sigma C_\alpha(\Delta\eta_\alpha)_i=2\Sigma C_\alpha R^t_i\eta_{\alpha t}=4n\Sigma C_\alpha \eta_{\alpha i}\ ,$$

by virtue of (2.1) and (2.9). Hence u is a harmonic 1-form because of $(\Delta u, u)=0$. Thus we have $C_\alpha=0$ and obtain the lemma.

LEMMA 3.2. *In a compact framed S-manifold,* $\tilde{w}=fw$ *is a harmonic 1-form for any harmonic 1-form w.*

PROOF. Taking account of lemma 3.1 and (2.6) we get

$$\begin{aligned}(\Delta\tilde{w})_j&=\nabla^a\nabla_a(f_j^{\ i}w_i)-R_j^{\ a}(f_a^{\ i}w_i)\\ &=(\nabla^a\Delta_a f_j^{\ i})w_i+2(\nabla_a f_j^{\ i})(\nabla^a w_i)+f_j^{\ i}\nabla^a\nabla_a w_i-f_j^{\ a}R_a^{\ i}w_i\ ,\end{aligned}$$

the first two terms of the right member is transformed to

$$-2sf_j^{\ i}w_i+2sf_j^{\ i}w_i+2s\Sigma\eta_{\alpha j}\xi_\beta^{\ \alpha}f_a^{\ i}w_i+2s\Sigma\eta_{\alpha j}\xi_\beta^{\ a}f_a^{\ i}w_i=0\ ,$$

by birture of (1.2) and (2.6). Thus we have

$$(\Delta\tilde{w})_j=f_j^{\ i}(\nabla^a\nabla_a w_i-R_i^{\ a}w_a)=0.$$

From lemmas 3.1 and 3.2 we have

THEOREM 3.3. *The first Betti number of a compact framed S-manifold is zero and even.*

Next, we define an f-analytic form as a 1-form u satisfying

$$fdu-dfu=0, \tag{3.2}$$

and

(3.3) $$i(\xi_\alpha)u=0 \quad \text{for all } \xi_\alpha .$$

Then the equation (3.2) is written explicitly as follows

(3.4) $$f_j^{\ a}(\nabla_a u_i-\nabla_i u_a)-\nabla_j(f_i^{\ a}u_a)-\nabla_i(f_j^{\ a}u_a)=0$$

Taking account of (3.3) and (2.5), the last equation is written by

(3.5) $$f_j^{\ a}\nabla_a u_i-f_i^{\ a}\nabla_j u_a+\Sigma_\alpha\eta_{\alpha j}u_i-\Sigma_\alpha\eta_{\alpha i}u_j=0 .$$

Transvecting (3.5) with g^{ji} we obtain

(3.6) $$f^{ji}\nabla_j u_i=0.$$

Then we prove

THEOREM 3.4. *A necessary and sufficient condition for a* 1*-form u in a compact framed S-manifold to be harmonic is that it is f-analytic.*

PROOF. For a harmonic 1-form u, we have $du=0$ and $i(\xi_\alpha)=0$ by virtue of lemma 3.1. Then fu is also a harmonic and we have $dfu=0$. Hence u is an f-analytic.

Conversely, let u be an f-analytic, then we have $i(\xi_\alpha)u=\xi_\alpha^{\ i}u_i=0$. Differenting the above and making use of (2.4) we get

$$f_j^{\ i}u_i+(\nabla_j u_i)\xi_\alpha^{\ i}=0.$$

Again diffirentiating the last equation and using of (3.6) we have

(3.7) $$\xi_\alpha^{\ i}\nabla^j\nabla_j u_i=0.$$

Next, transvecting (3.5) with $f_k^{\ i}$ we have

$$f_j^a f_k^{\ i}\nabla_a u_i+\nabla_j u_k-\Sigma\eta_{\alpha k}\xi_\alpha^a\nabla_j u_a+f_k^{\ i}\Sigma\xi_{aj}u_i=0.$$

Operating $\nabla^j=g^{ji}\nabla_i$ to the last equation we get $\Delta u=0$, by virtue of (3.7) and (2.5). Thus u is harmonic.

4. Harmonic 1-forms and Ricci curvature tensors

In this section, we use $2n$-homothetic deformations to get results on the first Betti numbers. First we put D by the equations $\eta_\alpha=0$ for all α, then D is a $2n$-dimensional distribution. We define a $2n$-homothetic deformation, or simply a D-homothetic deformation $g_{ji}\to{}^*g_{ji}$ is given by

(4.1) $$^*g_{ji} = ag_{ji}+b\Sigma\,\eta_{\alpha j}\eta_{\alpha i} ,$$

for the constants a and b satisfying $a>0$ and $a+b>0$, The inverse matrix

$(^*g^{jk})$ of $(^*g_{jk})$ is given by

$$(4.2)\qquad {}^*g^{jk}=a^{-1}g^{jk}-a^{-1}b(a+b)^{-1}\Sigma\,\xi^j_\alpha\xi^k_\alpha\,.$$

If we put

$$W^i_{jk}={}^*\Gamma^i_{jk}-\Gamma^i_{jk}\,,$$

we have in a framed S-manifold

$$(4.3)\qquad W^i_{jk}=-a^{-1}b\Sigma_\alpha(f^i_j\eta_{\alpha k}+f^i_k\eta_{\alpha j})\,.$$

Substituting (4.3) in to the

$$ {}^*R^i_{jkh}=R^i_{jkh}+\nabla_iW^h_{jk}-\nabla_jW^h_{ik}+W^h_{ai}W^a_{jk}-W^h_{aj}W^a_{ik}\,,$$

we have

$$\begin{aligned}(4.4)\qquad {}^*R^h_{ijk}=&R^h_{ijk}+a^{-1}bs(2f^h_kf_{ij}+f^h_jf_{ik}-f^h_if_{jk})\\&+a^{-1}b\;\Sigma(\eta_{\alpha k}\nabla_if^h_j+\eta_{\alpha j}\nabla_if^h_k-\eta_{\alpha k}\nabla_jf^h_i-\eta_{\alpha i}\nabla_jf^h_k)\\&+a^{-2}b^2(\delta^h_i\Sigma\eta_{\alpha j}-\delta^h_j\Sigma\eta_{\alpha i})\Sigma\eta_{\alpha k}\\&+a^{-2}b^2(\Sigma\xi^h_\alpha(\eta_{\alpha j}\Sigma\eta_{\beta i}-\eta_{\alpha i}\Sigma\eta_{\beta j})\Sigma\eta_{\gamma k}\end{aligned}$$

Contracting with respect to i and h we get

$$(4.5)\qquad {}^*R_{jk}=R_{jk}-2a^{-1}bsg_{jk}+2a^{-1}b(2n+s)\Sigma\eta_{\alpha j}\eta_{\alpha k}+2na^{-2}b^2\Sigma\eta_{\alpha j}\Sigma\eta_{\beta k}$$

where we have used (2.6). Contracting the last equation with (4.2), we have

$$(4.6)\qquad {}^*R=a^{-1}R-2nsa^{-2}b,$$

where R is the scalar curvature.

LEMMA 4.1. *For a framed manifold M with structure tensors $(f,g,\xi_\alpha,\eta_\alpha)$, we put*

$$(4.7)\qquad {}^*f=f,\ \ {}^*\xi_\alpha=a\xi_\alpha,\ \ {}^*\eta_\alpha=a^{-1}\eta_\alpha,\ \ {}^*g=ag+(a^2-a)\Sigma\eta_\alpha\otimes\eta_\alpha$$

for positive constant a. If $(f,g,\xi_\alpha,\eta_\alpha)$ is a framed S-structure (C-structure, resp.), then $({}^*f,{}^*g,{}^*\xi_\alpha,{}^*\eta_\alpha)$ is also a framed S-structure (C-structure resp.)

PROOF. By the definition it is easy to see that $({}^*f,{}^*g,{}^*\xi_\alpha,{}^*\eta_\alpha)$ is a framed metric structure. We compute

$$\begin{aligned}{}^*d{}^*\eta_\alpha(X,Y){}^*\xi_\alpha&={}^*d\eta_\alpha(X,Y)\xi_\alpha=({}^*\nabla_X\eta_\alpha)(Y)-({}^*\nabla Y\eta_\alpha)(X)\\&=(\nabla_X\eta_\alpha)(Y)-(\nabla_Y\eta_\alpha)(X)-d\eta_\alpha(X,Y)\xi_\alpha\end{aligned}$$

from which and ${}^*f=f$, we have

$$[{}^*f, {}^*f] + \Sigma \; {}^*d{}^*\eta_\alpha(X, Y){}^*\xi_\alpha = 0 .$$

From $F^*(X, Y) = aF(X, Y)$ we get ${}^*d{}^*F = adF = 0$. Thus the structure $(f^*, {}^*g, {}^*\xi_\alpha, {}^*\eta_\alpha)$ is a framed K-structure.

Furthermore we have

$$ {}^*d{}^*\eta_\alpha(X, Y) = ad\eta_\alpha(X, Y) = 2{}^*F(X, Y) ,$$

this shows that $({}^*f, {}^*g, {}^*\xi_\alpha, {}^*\eta_\alpha)$ is a framed S-structure.

LEMMA 4.2. *A harmonic* 1-*form* w *with respect to* g *on a framed S-manifold* M *is also a harmonic* 1-*form with respect to* ${}^*g$.

PROOF. Since $dw=0$ and $\delta w=0$, we prove ${}^*dw=0$ and ${}^*\delta w=0$. By the definitions of ${}^*d$ and ${}^*\delta$ we get

$$\begin{aligned}({}^*dw)_{ji} &= {}^*\nabla_j w_i - {}^*\nabla_i w_j \\ &= (\nabla_j w_i - W^a{}_{ij} w^a) - (\nabla_i w_j - W^a{}_{ji} w_a) .\end{aligned}$$

Since $W^a{}_{ij}$ is symmetric with respect to i and j, we have ${}^*dw=0$.

$$\begin{aligned}{}^*\delta w &= {}^*g^{ij}({}^*\nabla_j w_i) \\ &= (a^{-1}g^{ij} - a^{-1}b(a+b)^{-1}\xi^i_\alpha \xi^j_\alpha)(\nabla_j w_i - W^a_{ji}\, w_a) \\ &= a^{-1}\delta w .\end{aligned}$$

Hence we have ${}^*\delta w=0$.

THEOREM 4.3. *On a compact framed S-manifold* M, *there exists no harmonic* 1-*form* w *which satisfies*

$$R_1(w, w) + 2sg(w, w) > 0 \tag{4.8}$$

for any point of M *and which has at least one point where inquality holds. Especially, if* $R_1 + 2s\, g$ *is positive definite, then the first Beti number is zero, that is,* $b_1(M)=0$.

PROOF. Assume that there exists a harmonic 1-form w satisfying (4.8). As M is compact $g(w, w)$ is bounded and there exists a positive number ε such that

$$R_1(w, w) + 2sg(w, w) > \varepsilon > 0$$

holds everywhere over M.

On the other hand, by lemma 3.1, (4.2) and (4.5) we have

$$\begin{aligned}{}^*R_{ji}{}^*w^j{}^*w^i &= a^{-1}{}^*R_{ji}w^j w^i \\ &= a^{-1}(R_{ji} - 2a^{-1}bs\, g_{ji})w^j w^i .\end{aligned}$$

If we choose the constant a so small that

$$2asg(w,w)<\varepsilon,$$

then we have

$$*R_1(*w,*w)>0,$$

the last inquality contradicts to the Theorem of Yano and Bochner [9] .

5. Harmonic 1-forms and curvature tensors

In a framed manifold M, we define an f-basis at a point of M as the set of orthogonal frame $\{e_\lambda, e_\lambda{}^*, e_{\alpha'}\}$ $(\lambda^*=n+\lambda, \alpha'=2n+\alpha)$ such tha

(5.1) $$e_{\lambda^*}=fe_\lambda, \qquad e_{\alpha'}=\xi_\alpha$$

Then the components of the metric tensor g and the fundamental 2-form F with respect to an f-basis are given by

(5.2) $$g=\begin{pmatrix} \delta^\lambda_\mu & 0 & 0 \\ 0 & \delta^\lambda_\mu & 0 \\ 0 & 0 & \delta^\alpha_\beta \end{pmatrix} \qquad F=\begin{pmatrix} 0 & -\delta^\lambda_\mu & 0 \\ \delta^\lambda_\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. Then we have $f_{\lambda\lambda^*}=-1$, $f_{\lambda^*\lambda}=1$ and other components are all zero.

In a framed S-manifold, from (2.11), for an eigenvector X of R_1, fX is also an eigenvector. Thus we have an f-basis for which only $R_{\lambda\lambda}$, $R_{\lambda^*\lambda^*}$ and $R_{\alpha\alpha}=2n$ may be non-vanishing components of R_1. Hence the matrix (R_{ij}) is a diagonal.

By $K(X,Y)$ we mean the sectional curvature for the 2-plane determined by X and Y, and we put

$$K_{\lambda\mu}=K(e_\lambda,e_\mu), \qquad K_{\lambda\mu^*}=K(e_\lambda,e_{\mu^*}),$$
$$K_{\lambda\alpha}=K(e_\lambda,\xi_\alpha), \qquad K_{\lambda^*\alpha}=K(e_{\lambda^*},\xi_\alpha),$$

then we have

(5.3) $$K_{\lambda\mu}=K_{\lambda^*\mu^*}, \qquad K_{\lambda\mu^*}=K_{\lambda^*\mu},$$
$$K_{\alpha\beta}=0, \qquad K_{\lambda\alpha}=K_{\lambda^*\alpha}=1.$$

From (5.3) we get

(5.4) $$R_{\lambda\lambda}= s+\Sigma_\mu(K_{\lambda\mu}+K_{\lambda\mu^*}),$$

(5.5) $$R_{\lambda^*\lambda^*}= s+\Sigma_\mu(K_{\lambda^*\mu}+K_{\lambda^*\mu^*}).$$

THEOREM 5.1. *Let M be a compact framed S-manifold of dimension $2n+s$. If the sectional curvature of M satisfies the relation*

(5.6) $$\Sigma_\mu(K_{\lambda\mu}+K_{\lambda\mu^*})>-3s ,$$

then $b_1(M)=0$.

PROOF. For any vector $X=(a_\lambda, b_\lambda, 0)$ with respect to the f-basis $\{e_\lambda, e_{\lambda*}, \xi_\alpha\}$, we have

$$g(X, X)=\Sigma_\lambda(a_\lambda^2+b_\lambda^2)$$

and

$$R_1(X, X)=\Sigma R_{\lambda\lambda}(a_\lambda)^2+\Sigma R_{\lambda*\lambda*}(b_\lambda)^2,$$

substituting (5.4) and (5.5) into the last equation, we get

$$R_1(X, X)+2sg(X, X)=(\Sigma_\mu(K_{\lambda\mu}+K_{\lambda\mu*})+s)g(X, X)+2sg(X, X).$$

By hypothesis we see that

$$R_1(X, X)+2sg(X, X)>0 .$$

Now, suppose that w be a non-zero harmonic 1-form, then the vector X associated to w is orthogonal to ξ_α. This contradictory to Theorem 4.3. Hence w has to be zero and $b_1(M)=0$.

6. Harmonic 1-forms and f-holomorphic pinching

In a framed S-manifold, analogously to the Sasakian case [8], we define certain pinching for f-sectional curvature and discuss the relations of harmonic 1-forms and such a pinching. To get the relations we consider a D-homothetic deformation:

$$*g=ag+(a^2-a)\Sigma\eta_\alpha\otimes\eta_\alpha . \tag{6.1}$$

LEMMA 6.1. *For a D-homothetic deformation* (6.1) *on a framed S-manifold M, we have*

$$*K_{\lambda\mu}=a^{-1}K_{\lambda\mu}, \tag{6.2}$$

$$*K_{\lambda\mu*}=a^{-1}[K_{\lambda\mu*}+3s(1-a)\delta_{\lambda\mu}] \tag{6.3}$$

$$(\textit{especially, } *K_{\lambda\lambda*}+3s=a^{-1}(K_{\lambda\lambda*}+3s))$$

PROOF. For an f-basis $(e_\lambda, e_{\lambda*}, \xi_\alpha)$, the related $*f$-basis is given by

$$*e_\lambda=a^{-1/2}e_\lambda,\quad *e_{\lambda*}=a^{-1/2}e_{\lambda*},\quad *\xi_\alpha=a^{-1}\xi_\alpha .$$

Let X and Y be orthonormal vectors with respect to g in D, where D is the distribution defined by $\eta_\alpha=0$, From (4.1) and (4.3) we have

$$\begin{aligned}*K(X, Y)&=*g(*R(X, Y)X, Y)/*g(X, X)*g(Y, Y)\\ &=a^{-1}[K(X, Y)+3s(1-a)F(X, Y)^2]\end{aligned}$$

Since $F(e_\lambda, e_\mu)=0$ and $F(e_\lambda, e_{\mu^*})=-\delta_{\lambda\mu}$, we have (6.2) and (6.3).

Now assumme that H and L defined by

$$H=\sup\{K(X,fX)\};\ X\in D,$$
$$L=\inf\{K(X,fX)\};\ X\in D,$$

exist and that $H+3s>0$, then t defined by

$$t=(L+3s)/(H+3s) \tag{6.4}$$

is invariant for the D-homothetic deformation (6.1). In this case we say that M is f-holomorphically pinched.

LEMMA 6.2. *If a framed S-manifold M is f-holomorphically pinched, we can find a Riemannianetric* $*g$ *by D-homothetic deformation so that* $*H=s$ *and* $*L=(4t-3)s$ *with respect to* $(f, *g, *\xi_\alpha, *\eta_\alpha)$.

PROOF. If we put $a=(H+3s)/4s$, then from (6.3) we have $*H=s$.

LEMMA. 6.3. (D.E. Blair [1]) *Let M be a framed S-manifold, then for any vectors* $X, Y \in D$, *we have*

$$g(R(X,Y)X,Y)=\frac{1}{32}[3D(X+fY)+3D(X-fY)-D(X+Y)-D(X-Y)-4D(X)-4D(Y)-24sP(X,Y)], \tag{6.5}$$

where $D(X)=g(R(X,fX)X,fX)$ *and*

$$P(X,Y)=g(X,Y)^2-g(X,X)g(Y,Y)+F(X,Y)^2.$$

Especially if X and Y are orthonormal, denoting $H(X)=K(X,fX)$ *and* $g(X,fY)=\cos\theta$, *we have*

$$K(X,Y)=\frac{1}{8}[3(1+\cos\theta)^2H(X+fY)+3(1-\cos\theta)^2H(X-fY)-H(X+Y)-H(X-Y)-H(X)-H(Y)+6s\sin^2\theta] \tag{6.6}$$

LEMMA 6.4. *In a framed S-manifold M, for an orthonormal pair* $X, Y\in D$, *we have*

$$K(X,Y)+\sin^2\theta\, K(X,fX)=\frac{1}{4}[(1+\cos\theta)^2H(X+fY)+(1-\cos\theta)^2H(X-fY)+H(X+Y)+H(X-Y)-H(X)-H(Y)+6s\sin^2\theta] \tag{6.7}$$

PROOF. Replacing Y by fY in (6.5) and adding the resulting equation to (6.6), we get (6.7).

Finally we prove

THEOREM 6.5. *Let M be a framed S-manifold which is f-holomorphically pinched with* $t>\frac{1}{2}\left(1-\frac{1}{n}\right)$. *Then* $b_1(M)=0$.

PROOF. We put $X=e_\lambda$ and $Y=e_\mu$ in (6.7), then(6,7) is written by

$$K_{\lambda\mu}+K_{\lambda\mu*}=\frac{1}{4}[H(e_\lambda+e_{\mu_*})+H(e_\lambda-e_{\mu*})+H(e_\lambda+e_\mu)$$
$$+H(e_\lambda-e_\mu)-H(e_\lambda)-H(e_\mu)+6s] \ .$$

By a D-homothetic deformation (6.1), the last equation is transformed into

$$a({}^*K_{\lambda\mu}+{}^*K_{\lambda\mu*})=\frac{a}{4}[{}^*H(e_\lambda+e_{\mu*})+{}^*H(e_\lambda-e_{\mu*})+{}^*H(e_\lambda+e_\mu)$$
$$+{}^*H(e-e_\mu)-{}^*H(e_\lambda)-{}^*H(e_\mu)+6s] \ ,$$

from which and lemma 6.2 we have

$$4st-2s\leq{}^*K_{\lambda\mu}+{}^*K_{\lambda\mu*}\leq 4s-2st$$

(6.8) $$\Sigma_\mu({}^*K_{\lambda\mu}+{}^*K_{\lambda\mu*})=\Sigma_{\lambda\neq\mu}({}^*K_{\lambda\mu}+{}^*K_{\lambda\mu*})+{}^*K_{\lambda\lambda*}$$
$$\geq(4nt-2n-1)s$$

Therefore, by Theorem 4.3 we have $t>(n-1)/2n$ for $s>0$.

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