Kyungpook Math. J.
Volume 15, Number 2
December, 1975

PERFECT MAPS ON THE c-CONTINUOUS FUNDAMENTAL GROUP

By Jin Ho Kwak

1. Introduction

In (2), Gentry and Hoyle introduced the concept of c-continuous fundamental group using c-continuous homotopy for the equivalence relation instead of the usual homotopy. The goal of this paper is to find the functor from the category of pointed spaces and perfect maps to the category of groups which assign to a pointed space its c-continuous fundamental group, and to find the effect of c-continuous homotopy on the c-continuous fundamental group. Throughout this paper the symbol I used to denote the closed interval [0, 1], and the symbol $f\sim g$ $(f\underset{y_0}{\sim}g)$ will mean that f is homotopic to g (relative y_0). The reader is referred to [3], [4] for definitions and notations not covered in this paper.

2. Definitions and preliminaries

DEFINITION 2.1. [1] Let X and Y be topological spaces, let $f:X\longrightarrow Y$ be a map, and let $p\in X$, then f is said to be *c-continuous* at p provided if U is an open subset of Y containing $f(p)$ such that $Y-U$ is compact, then there is an open subset V of X containing p such that $f(V)\subset U$. The function f is said to be *c-continuous* on X provided f is c-continuous at each point of X.

DEFINITION 2.2. [2] Let X and Y be a topological spaces, and let f, $g:X\longrightarrow Y$ be continuous maps (and $A\subset X$). We say that f is *c-continuous homotopy* to g (relative A), denoted by $H:f\overset{c}{\sim}g$, briefly $f\overset{c}{\sim}g(H:f\underset{A}{\overset{c}{\sim}}g, f\underset{A}{\overset{c}{\sim}}g)$ provided there is a c-continuous map $H:X\times I\longrightarrow Y$ such that $H(x,0)=f(x)$, $H(x,1)=g(x)$, (and $H(a,t)=f(a)=g(a)$) for all $x\in X$, $t\in I$ $(a\in A)$.

Let Y be a topological space, $y_0\in Y$ and let $C(Y,y_0)$ be the set of all continuous maps $\alpha:I\longrightarrow Y$ such that $\alpha(0)=y_0=\alpha(1)$.

The relation $\underset{y_0}{\overset{c}{\sim}}$ is an equivalence relation on $C(Y,y_0)$, and the equivalence class of α is denoted by $[\alpha]$ and $C_1(Y,y_0)$ denotes the set of all equivalence classes.

DEFINITION 2.3. [2] Let $\alpha, \beta \in C(Y, y_0)$. Then $\alpha * \beta$ is the map in $C(Y, y_0)$ defined by

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

and we define $[\alpha] \cdot [\beta] = [\alpha * \beta]$ in $C_1(Y, y_0)$.

THEOREM 2.4 [2] *The ordered pair* $(C_1(Y, y_0), \cdot)$ *is a group,* which is called the c-continuous fundamental group.

DEFINITION 2.5. Let X and Y be topological spaces. A map $f: X \longrightarrow Y$ is called *perfect* if it is continuous closed (not necessary surjective) and each fiber $f^{-1}(y)$, $(y \in Y)$ is compact.

It is well-known that the inverse image of a compact set under a surjective perfect map is compact. From this, we can easily have the following theorem.

THEOREM 2.6. *If X and Y are topological spaces, $f: X \longrightarrow Y$ is a perfect map, and K is a compact subset of Y, then the inverse image $f^{-1}(K)$ of K is a compact subset of X.*

From this, we have

COROLLARY 2.7. *If X, Y and Z are topological spaces, and maps $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ are perfect, then the composite map $g \circ f: X \longrightarrow Z$ is perfect.*

In [1, Example 3], we show that the composite map $g \circ f$ of c-continuous map f and continuous map g is not generally c-continuous map but we have the

COROLLARY 2.8. *Let X, Y and Z be topological spaces, $f: X \longrightarrow Y$ be a c-continuous, and let $g: Y \longrightarrow Z$ be a perfect map. Then the composite map $g \circ f: X \longrightarrow Z$ is a c-continuous map.*

PROOF. Let U be open subset of Z with compact compliment. Then since g is perfect, $g^{-1}(U)$ is open and $Y - g^{-1}(U) = g^{-1}(Z-U)$ is compact by Theorem 2.6, hence, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open, for f is c-continuous.

3. Main results

LEMMA 3.1. *Every perfect map $f: (X, x_0) \longrightarrow (Y, y_0)$ induces a homomorphism $f_\#: C_1(X, x_0) \longrightarrow C_1(Y, y_0)$. In fact, the induced homomorphism $f_\#: C_1(X, x_0) \longrightarrow C_1(Y, y_0)$ is defined by $f_\#([\alpha]) = [f \circ \alpha]$.*

PROOF. Let $[\alpha] \in C_1(X, x_0)$. Then $\alpha: I \longrightarrow X$ is a continuous map such that

$\alpha(0)=x_0=\alpha(1)$, hence, the composite map $f\circ\alpha: I\longrightarrow Y$ is continuous and $(f\circ\alpha)(0)=y_0=(f\circ\alpha)(1)$, i.e. $[f\circ\alpha]\in C_1(Y,y_0)$. And let F: $\alpha\underset{x_0}{\overset{c}{\sim}}\beta$, then, we easily have that $f\circ F$: $f\circ\alpha\underset{y_0}{\overset{c}{\sim}} f\circ\beta$ by Corollary 2.8. Therefore, the map $f_\#$ is well-defined on $C_1(X,x_0)$. Now, it remains to show that $f_\#$ is a homomorphism. Let $[\alpha]$, $[\beta]$ be arbitrarily given in $C_1(X,x_0)$, then $f_\#([\alpha]\cdot[\beta])=f_\#([\alpha*\beta])=[f\circ(\alpha*\beta)]=[(f\circ\alpha)*(f\circ\beta)]=[f\circ\alpha]\cdot[f\circ\beta]=f_\#([\alpha])\cdot f_\#([\beta])$, which proves the remain part.

LEMMA 3.2. *Let* $f: X\longrightarrow Y$, $g: Y\longrightarrow Z$ *be perfect maps, then the induced homomorphism* $(g\circ f)_\#$, $g_\#\circ f_\#: C_1(X,x)\longrightarrow C_1(Z,(g\circ f)(x))$ *are equal.*

PROOF. By Corollary 2.7, $(g\circ f)_\#$ is well-defined, and $(g\circ f)_\#([\alpha])=[g\circ f\circ\alpha]=g_\#([f\circ\alpha])=(g_\#\circ f_\#)([\alpha])$, for all $[\alpha]\in C_1(X,x)$.

From Lemma 3.1 and Lemma 3.2, we have the

THEOREM 3.3. *There is a covariant functor from the category of pointed spaces and perfect maps to the category of groups which assign to a pointed space its c-continuous fundamental group and to a perfect map f the induced homomorphism* $f_\#$.

PROOF. It is clear that if f is the identity on a topological space X, $f_\#$ is the identity on $C_1(X,x)$.

COROLLARY 3.4. *Let X and Y be topological spaces. If* $f: X\longrightarrow Y$ *is a homeomorphism, then the induced homomorphism* $f_\#: C_1(X,x)\longrightarrow C_1(Y,f(x))$ *is an isomorphism.*

PROOF. Since $(f^{-1})_\#\circ f_\#=(f^{-1}\circ f)_\#=(1_X)_\#=1_{C_1(X,x)}$, $f_\#\circ(f^{-1})_\#=(f\circ f^{-1})_\#=(1_Y)_\#=1_{C_1(Y,f(x))}$, $f_\#$ is monomorphism and epimorphism, hence, $f_\#$ is isomorphism.

Next, we will investigate the dependence of the group $C_1(X,x)$ on the base point x. Let x and y be two points of X, and let γ be a path in X from x to y, then, by the Theorem 6 of [2], $C_1(X,x)$ and $C_1(X,y)$ are isomorphic, by the isomorphism $[\alpha]\longrightarrow[\gamma^{-1}*\alpha*\gamma]$, $([\alpha]\in C_1(X,x))$. Furthermore we have the

THEOREM 3.5. *Let X and Y be topological spaces,* $f: X\longrightarrow Y$ *be perfect map, and let* γ *be a path in X from* x_0 *to* x_1. *Then the diagram*

$$\begin{array}{ccc} C_1(X,x_0) & \xrightarrow{f_\#} & C_1(Y,f(x_0)) \\ \downarrow u & & \downarrow v \\ C_1(X,x_1) & \xrightarrow{f_\#} & C_1(Y,f(x_1)) \end{array}$$

is commutative,

where $u: C_1(X, x_0) \longrightarrow C_1(X, x_1)$ *is isomorphism by the formula* $u([\alpha]) = [\gamma^{-1} * \alpha * \gamma]$, *for all* $[\alpha] \in C_1(X, x_0)$, $v: C_1(Y, f(x_0)) \longrightarrow C_1(Y, f(x_1))$ *is isomorphism by the formula* $v([\beta]) = [(f \circ \gamma)^{-1} * \beta * (f \circ \gamma)]$, *for all* $[\beta] \in C_1(Y, f(x_0))$.

PROOF. Since γ is a path in X from x_0 to x_1, $f \circ \gamma$ is a path in Y from $f(x_0)$ to $f(x_1)$, hence the isomorphism $v: C_1(Y, f(x_0)) \longrightarrow C_1(Y, f(x_1))$ is well definded. And, for each $[\alpha] \in C_1(X, x_0)$, $(v \circ f_\#)([\alpha]) = v([f \circ \alpha]) = [(f \circ \gamma)^{-1} * (f \circ \alpha) * (f \circ \gamma)] = [f \circ (\gamma^{-1} * \alpha * \gamma)] = f_\#([\gamma^{-1} * \alpha * \gamma]) = (f_\# \circ u)([\alpha])$.

THEOREM 3.6. *Let* X *and* Y *be topological spaces,* $x_0 \in X$, *and let* $f, g: X \longrightarrow Y$ *be perfect maps such that* f *is c-continuous homotopic to* g *relative to* x_0. *Then the induced homomorphism* $f_\#$, $g_\#: C_1(X, x_0) \longrightarrow C_1(Y, f(x_0))$ *are the same.*

PROOF. Let $F: X \times I \longrightarrow Y$ be a c-continuous map such that

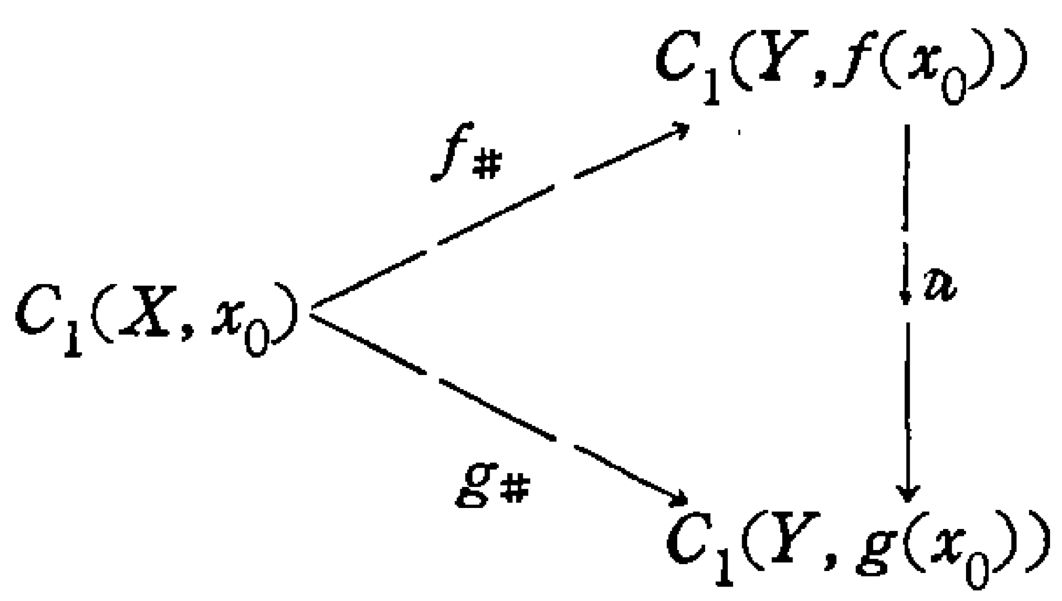
$$F(x, 0) = f(x),$$
$$F(x, 1) = g(x),$$
$$F(x_0, t) = f(x_0) = g(x_0)$$

for all $x \in X$, $t \in I$.
For an arbitrarily given $[\alpha]$ in $C_1(X, x_0)$, we define $H: I \times I \longrightarrow Y$ by $H(s, t) = F(\alpha(s), t)$, then, since $H = F \circ (\alpha \times 1_I)$, H is a c-continuous map by [1. Th. 3] and

$$H(s, 0) = F(\alpha(s), 0) = (f \circ \alpha)(s)$$
$$H(s, 1) = F(\alpha(s), 1) = (g \circ \alpha)(s)$$

for all $s \in I$.
Hence, $f_\#([\alpha]) = g_\#([\alpha])$ for all $[\alpha] \in C_1(X, x_0)$.

THEOREM 3.7. *Let* X *and* Y *be topological spaces,* $f, g: X \longrightarrow Y$ *be perfect maps, and let* $F: X \times I \longrightarrow Y$ *be a homotopy between* f *and* g. *Choose a base point* $x_0 \in X$, *and let* γ *be the path given by* $\gamma(t) = F(x_0, t)$, $0 \leq t \leq 1$. *Then the following diagram is commutative:*



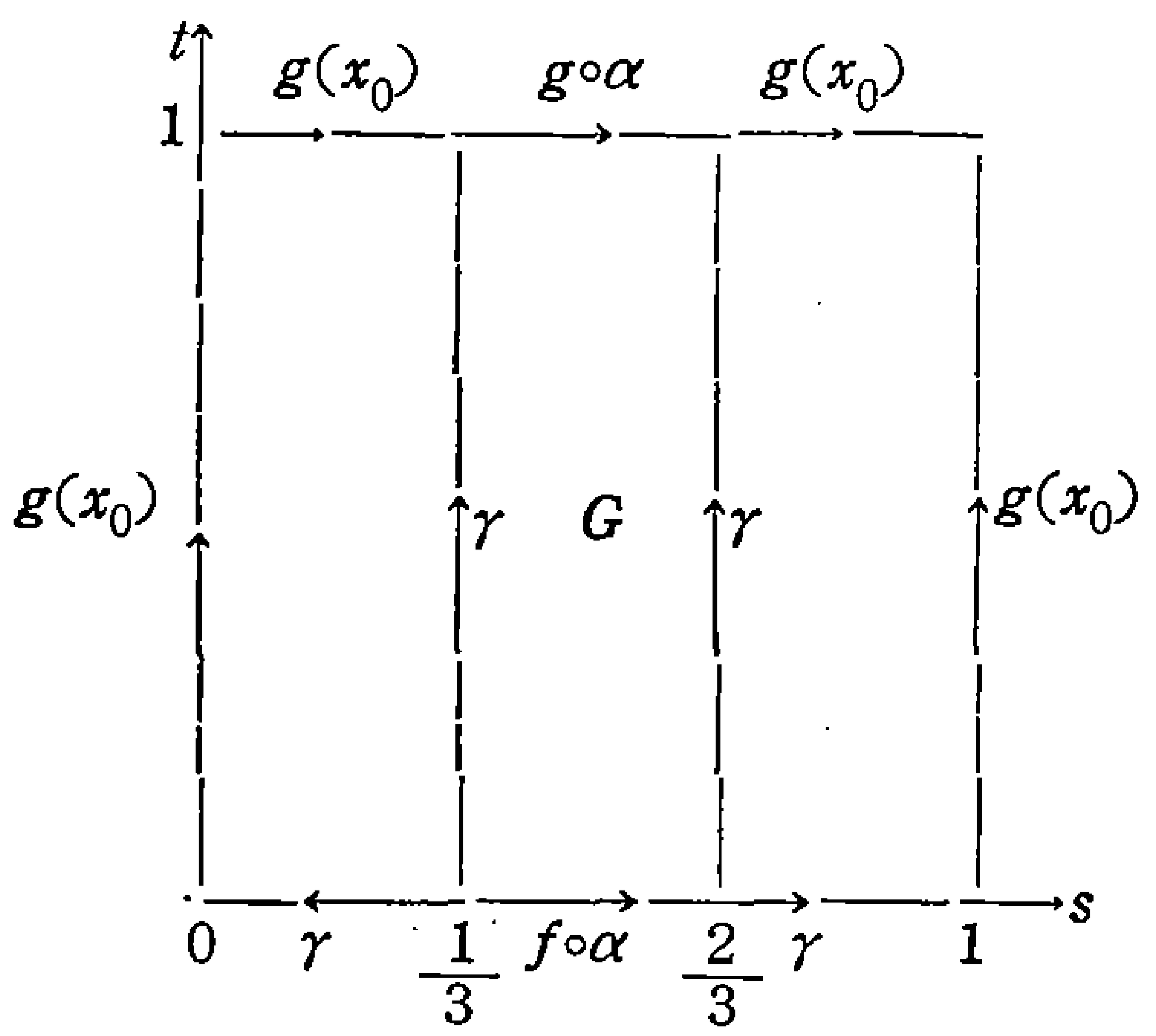
where $u;C_1(Y,f(x_0))\longrightarrow C_1(Y,g(x_0))$ *is isomorphism defined by the formula* $u([\beta])=[\gamma^{-1}*\beta*\gamma]$, *for all* $[\beta]\in C_1(Y,f(x_0))$. *Furthermore,* $f_{\#}$ *is isomorphism iff* $g_{\#}$ *is isomorphism.*

PROOF. Let $[\alpha]\in C_1(X,x_0)$. Consider the map $G:I\times I\longrightarrow Y$ defined by $G(s,t)=F(\alpha(s),t)$, for all $s,t\in I$. Then, G is a continuous map, and for $s,t\in I$, we have

$$G(s,0)=F(\alpha(s),0)=(f\circ\alpha)(s)$$
$$G(s,1)=F(\alpha(s),1)=(g\circ\alpha)(s)$$
$$G(0,t)=F(x_0,t)=\gamma(t)=G(1,t).$$

Now, we define a map $H:I\times I\longrightarrow Y$ by

$$H(s,t)=\begin{cases} g(x_0) & ;3s\leq t,\ 0\leq s\leq 1/3 \\ \gamma(1+t-3s) & ;3s\geq t,\ 0\leq s\leq 1/3 \\ G(3s-1,t) & ;1/3\leq s\leq 2/3 \\ g(x_0) & ;-3s+3\leq t,\ 2/3\leq s\leq 1 \\ \gamma(3s+t-2) & ;-3s+3\geq t,\ 2/3\leq s\leq 1. \end{cases}$$



Then, H is continuous on $I\times I$. Hence we easily have that

$$g\circ\alpha \underset{g(x_0)}{\simeq} \gamma^{-1}*(f\circ\alpha)*\gamma \text{ i.e., } g_{\#}([\alpha])=(u\circ f_{\#})([\alpha]),$$

which implies the commutativity of the diagram.

THEOREM 3.8. *If* X *and* Y *are topological spaces,* $f:X\longrightarrow Y$ *is a perfect map and a homotopy equivalence with a perfect homotopy inverse* $g:Y\longrightarrow X$, *then,* $_{\#}:C\,{}^1f(X,x)\longrightarrow C_1(Y,f(x))$ *is an isomorphism for any* $x\in X$.

PROOF. Because $g\circ f\sim$identity: $X\longrightarrow X$, by Theorem 3.7, we have the following commutative diagram:

$$\begin{array}{ccc} C_1(X,x) & \xrightarrow{(1_X)_\#} & C_1(X,x) \\ & \searrow^{(g\circ f)_\#} & \downarrow u \\ & & C_1(X,(g\circ f)(x)) \end{array}$$

where $u:C_1(X,x)\longrightarrow C_1(X,(g\circ f)(x))$ is an isomorphism induced by the path from x to $(g\circ f)(x)$. Since $(1_X)_\#$ is isomorphism, $g_\#\circ f_\#=(g\circ f)_\#$ is isomorphism, which implies that $f_\#$ is a monomorphism and $g_\#$ is an epimorphism. Similarly, we can prove that $f_\#$ is an epimorphism and $g_\#$ is a monomorphism from the fact $f\circ g\sim$identity: $Y\longrightarrow Y$. Hence, we conclude that $f_\#$ is an isomorphism.

Kyungpook University

REFERENCES

[1] K.R. Gentry and H.B. Hoyle, III, *c-continuous functions*, Yokohama Math. J. 18 (1970), pp. 71—76.

[2] ________, *c-continuous fundamental group*, Fund. Math. LXXVI (1972), pp. 9—17.

[3] W.S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, Inc. 1967.

[4] S. Willard, *General Topology*, Addison-Wesley Publ. Co. 1968.