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DISTANCE IN THE FINITE AFFINE PLANE

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1. Introduction

In 1960, J.W. Archbold [2] showed that a finite affine plane with coordinates in the finite field $GF(2^n)$ possesses a symmetric, rotationally and translationally invariant metric. Moreover, Archbold's metric d is "directed" in the sense that if P_1, P_2 and P_3 are collinear points, then $d(P_1, P_2)+d(P_2, P_3)=d(P_1, P_3)$.

D.W. Crowe showed in 1964 [3] that $GF(2^{2n})$ acts like an Argand diagram over $GF(2^n)$ and used this polar representation to derive the trigonometry of the finite affine plane over $GF(2^{2n})$.

It is well known that any affine plane π can be obtained by deleting one line L_∞ from an appropriate projective plane π^* and, moreover, that π^*(and hence π) can be coordinatized with elements from a system $(R, +, \cdot)$ where $(R, +, \cdot)$ is a double loop; i.e.,

(1) $(R, +)$ is a loop with identity 0.

(2) $(R-\{0\}, \cdot)$ is a loop with identity 1.

(3) For any $x\in R$, $0\cdot x=0=x\cdot 0$.

The purpose of this paper is to determine which finite affine planes possess a "metric" function d mapping the set of ordered pairs of points into the coordinatizing double loop R.

Perhaps the most basic properties inherent in the usual notion of distance are the following: Let $P_i(i=1,2,3,4)$ be affine points. Then (1) $d(P_1, P_2)=0$ if and only if $P_1=P_2$; (2) If there exists a finite sequence of translations $T_1, T_2, \cdots, T_n$ such that $T_1T_2\cdots T_n(P_1)=P_3$ and $T_1T_2\cdots T_n(P_2)=P_4$ then $d(P_1, P_2)=d(P_3, P_4)$; (3) If no such finite sequence of translations exists and P_1P_2 is parallel to P_3P_4 then $d(P_1, P_2)\neq d(P_3, P_4)$; We will show that every finite affine plane possesses a metric d with these properties and moreover, d is surjective (and hence directed) if and only if $(R, +, \cdot)$ is a right Veblen-Wedderburn system.

2. Existence of a metric

If L_1 and L_2 are lines of π^* concurrent with L_∞ at the point W and if $A(\neq W)$

is an ideal point then the perspectivity from L_1 to L_2 with center A will be called a *W-perspectivity*. A finite sequence of W-perspectivities is a *W-projectivity*.

For each ideal point W, let $\overline{W}$ denote the collection of all pairs (P, Q) such that P and Q are affine points and P, Q, W are collinear in π^*. If (P_1, Q_1), $(P_2, Q_2) \in \overline{W}$ then (P_1, Q_1) is *W-equivalent* to (P_2, Q_2) if there exists a W-projectivity T such that $T(P_1)=P_2$ and $T(Q_1)=Q_2$. Obviously W-equivalence is an equivalence relation on $\overline{W}$.

Let $\overline{\overline{W}}$ denote the set of W-equivalence classes. Evidently there exists a unique W-equivalence class N such that for any affine point P, $(P, P) \in N$. An injection $d_W : \overline{\overline{W}} \to R$ will be called a *W-metric* if $d_W(N)=0$.

Let $\Omega = \bigcup_{W I L_\infty} \overline{\overline{W}}$. A function $d : \Omega \to R$ will be called a *metric* if, for each ideal point W, the restriction d_W of d to $\overline{\overline{W}}$ is a W-metric.

THEOREM 1. *Every finite affine plane has a metric.*

PROOF. It is clearly sufficient to show that for each ideal point W, π has a W-metric. It is well known that there exists a positive integer b such that each line of π contains precisely b points. If B denotes a W-equivalence class and if L is a line of π containing W then B contains at least b distinct pairs (P, Q) where P and Q are incident with L. Since there are exactly $b-1$ lines parallel to L, then B contains at least b^2 elements. But the total number of elements of $\overline{W}$ is exactly b^3. Thus the number of W-equivalence classes cannot exceed b which is precisely the number of elements in R.

3. Existence of a directed W-metric

If d_W is a W-metric and P and Q are affine points such that P, Q, W are collinear in π^* then let $d_W(P, Q)=d_W(B)$ where B is the unique W-equivalence class containing (P, Q). We shall say that d_W is *directed* if for each affine point P and for each $r \in R$ there exists a unique affine point Q such that $d_W(P, Q)=r$.

LEMMA 1. *Let W be an ideal point such that π^* is (W, L_∞)-Desarguesian. If L_1, L_2, L_3 are distinct lines of π concurrent at W and if $T_i : L_i \to L_{i+1} (i=1, 2)$ are W-perspectivities then $T_2T_1 : L_1 \to L_3$ is a W-perspectivity.*

PROOF. If P and Q are distinct affine points of L_1 then the line $[T_2T_1(P)] \cdot P$ intersects the line $[T_2T_1(Q)] \cdot Q$ at the ideal point A. Thus T_2T_1 is the W-perspectivity with center A.

LEMMA 2. *Let W be an ideal point such that π^* is (W, L_∞)-Desarguesian. If $L_i (i=1, 2, 3, 4)$ are lines of π concurrent at W and if $T_j : L_j \to L_{j+1} (j=1, 2, 3)$ are W-perspectivities then $T_3T_2T_1$ is a W-perspectivity if $L_1 \neq L_4$.*

PROOF. If $L_1 \neq L_3$ or if $L_1 = L_2$, then the result is obvious. Hence we may assume that $L_3 = L_1 \neq L_2$. If $L_2 \neq L_4$ then Lemma 1 shows that T_3T_2 is a W-perspectivity and by the same lemma, $(T_3T_2)T_1$ is a W-perspectivity. Thus we may assume that $L_2 = L_4$:

Let P_1 and Q_1 be distinct points of L_1. Let $T_i(P_i) = P_{i+1}, T_i(Q_i) = Q_{i+1} (i=1, 2, 3)$. If Q_1Q_4 is parallel to Q_2Q_3, let $Q = Q_1Q_2 \cdot Q_3Q_4$ and let $P = QW \cdot P_3P_4$. Then triangles PP_3P_2 and QQ_3Q_2 are centrally perspective from W whence $P = P_3P_4 \cdot P_1P_2$. Hence triangles QQ_1Q_4 and PP_1P_4 are centrally perspective from W whence P_1P_4 is parallel to Q_1Q_4. Otherwise, if Q_1Q_4 is not parallel to Q_2Q_3, let $Q' = Q_1Q_4 \cdot Q_2Q_3$ and let $P' = QW \cdot P_2P_3$. As before, $P' = P_1P_4 \cdot P_3P_2$. Hence, in either case, P_1P_4 is parallel to Q_1Q_4 whence $T_3T_2T_1$ is the W-perspectivity with center $Q_1Q_4 \cdot P_1P_4$.

An easy induction now shows

LEMMA 3. *Let W be an ideal point such that π^* is (W, L_∞)-Desarguesian. If L_1 and L_2 are distinct lines of π and if $T : L_1 \to L_2$ is a W-projectivity then T is a W-perspectivity.*

THEOREM 2. *Let W be an ideal point of π. Then π^* is (W, L_∞)-Desarguesian if and only if π has a directed W-metric.*

PROOF. Assume that π^* is (W, L_∞)-Desarguesian. If (P, Q_1) is W-equivalent to (P, Q_2) it follows from Lemma 3 that $Q_1 = Q_2$. Thus $\overline{\overline{W}}$ contains precisely b elements where b is the number of elements in R. Thus any W-metric is bijective and hence directed.

Conversely, assume that π has a bijective W-metric. Let triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ be centrally perspective from W and assume that P_1P_2 is parallel to Q_1Q_2 and P_2P_3 is parallel to Q_2Q_3. Suppose, for sake of contradiction, that P_1P_3 is not parallel to Q_1Q_3. Let $A = L_\infty \cdot P_1P_3$ and let $T : P_3Q_3 \to P_1Q_1$ be the unique W-perspectivity with center A. Then (P_1, Q_1) is W-equivalent to $(P_1, T(Q_3))$ and $Q_1 \neq T(Q_3)$. Thus the number of W-equivalence classes is less than the number of distinct points on any line of π and this is the desired contradiction.

4. Existence of a directed metric

If d is a metric such that $d(P, Q)+d(Q, S)=d(P, S)$ whenever P, Q and S are collinear affine points, then d is *directed* if, for each ideal point W, the restriction d_W of d to $\overline{\overline{W}}$ is a directed W-metric.

LEMMA 4. *Let π^* be a right V-W plane. Let $P_i=(x_i, y_i)$, $Q_i=(z_i, w_i)$ $(i=1, 2)$ be four distinct points of π such that Q_1Q_2 is parallel to P_1P_2 and P_1Q_1 is parallel to P_2Q_2. Then $x_2-x_1=z_2-z_1$ and $y_2-y_1=w_2-w_1$.*

PROOF. We will show that $x_2-x_1=z_2-z_1$, the other result being obtained similarly. If $x_2=x_1$, the result is obvious. Hence we may let $P_1Q_1=[m, k_1]$, $P_2Q_2=[m, k_2]$, $P_1P_2=[n, k_3]$, $Q_1Q_2=[n, k_4]$ where $m, n, k_1, k_2, k_3, k_4 \in R$. Straightforward algebraic manipulation then shows that

$$(x_2-z_2-x_1+z_1)m=(x_2-z_2-x_1+z_1)n.$$

Since $m \neq n$ and $(R-\{0\}, \cdot)$ is a loop, it follows that $x_2-x_1=z_2-z_1$.

THEOREM 3. *π has a directed metric if and only if $(R, +, \cdot)$ is a right V-W system.*

PROOF. If π has a directed metric then the previous theorem shows that π^* is (W, L_∞)-Desarguesian for each ideal point W, whence $(R, +, \cdot)$ is a right V-W system.

Conversely, for each ideal point $W(\neq Y)$ let s_W be a non-zero element of R. Let $P_i=(x_i, y_i)$ $(i=1, 2)$. Then,

$$d(P_1, P_2)=(x_2-x_1)s_W \quad \text{if } (P_1, P_2) \in W$$

and

$$d(P_1, P_2)=y_2-y_1 \quad \text{if } (P_1, P_2) \in Y$$

defines a directed metric d by Lemma 4.

Some of these results appeared in the author's dissertation written at the Illinois Institute of Technology.

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