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THE SEMIGROUP OF ACYCLIC MATRICES

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Dedicated to Prof. C.K. Pahk on his 60th birthday

1. Introduction

Following A. Berman and A. Kotzig [1], we say that (0,1)-matrix is called an *acyclic matrix* if it does not have a permutation matrix of order two as a submatrix. Such matrices may be taken either over the two-element Boolean algebra {0,1} or over the field Z_2. Since a (0,1)-matrix over the Boolean algebra can be represented as a binary relation, it is advantageous to work with (0,1)-matrices over the Boolean algebra. Such matrices are called Boolean matrices.

Although the maximal order of cyclicity of acyclic matrices have been studied from the graph-theoretical point of view by A. Berman and A. Kotzig in [1], no work has been done on the semigroup properties of acyclic matrices. In doing so we obtained a partial solution to B.M. Schein's problem [4].

We use the Boolean matrix notation of [2] and the semigroup notation of [3].

$B_r(A)$ denotes the row basis of A,

$B_c(A)$ denotes the column basis of A,

$\rho_r(A)$ denotes the row rank of A,

$\rho_c(A)$ denotes the column rank of A,

$\rho(A)$ denotes the rank of A,

$R(A)$ denotes the row space of A,

$C(A)$ denotes the column space of A,

$|X|$ denotes the cardinality of a set X,

$E(X)$ denotes the set of all idempotents in a set X,

T_r denotes the $r\times r$ (0,1)-upper triangular matrix,

S_n denotes the set of all $n\times n$ permutation matrices,

$S(m,k)$ denotes the Stirling number of the second kind.

2. Preliminary results

Let A_n denote the set of all $n\times n$ acyclic matrices. Then A_n is a semigroup under matrix multiplication. We shall consider only nonzero matrices. It is clear that if A is an element of A_n, then $B_r(A)$ $(B_c(A))$ consist of set of all distinct rows (columns) of A. we will now state without proof some simple facts about acyclic matrices.

PROPOSITION 1. *If A is an element of A_n, then $\rho_r(A)=\rho_c(A)$.*

This proposition allows us to just talk about the rank of acyclic matrix, and so we will not distinguish between row and column rank and will denote both by ρ.

PROPOSITION 2. *A is an element of A_n iff $|R(A)|=\rho(A)+1=|C(A)|$, where* $1\leq|R(A)|\leq n+1$.

PROPOSITION 3. *Let U be an $n\times r$ acyclic matrix and V be an $r\times n$ acyclic matrix. Then UV is an element of A_n.*

PROPOSITION 4. *Let A be an element of A_n and $\rho(A)=r$. Let $B_r(A)=\{v_1,\cdots,v_r\}$ such that $v_i+v_{i+1}=v_i$ for every $i<r$ and $B_c(A)=\{u_1,\cdots,u_r\}$ such that $u_i+u_{i+1}=u_{i+1}$ for every $i<r$.*

Let $U=[u_1\cdots u_r]_{n\times r}$ *and* $V=\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix}_{r\times n}$.

Then $A^2=A$ iff $VU=T_r$.

PROOF. Necessity: Let $X=VU$. We show that $\rho_r(X)=r=\rho_c(X)$. We show only $\rho_c(X)=r$. We must have $R(A)=R(V)=R(VUV)$ but $C(VUV)\subset C(VU)\subset C(V)$. If $\rho_c(VU)<r$, then $|C(VU)|<r+1$ and so $|C(VUV)|\neq r+1$. This implies $|R(VUV)|\neq r+1$. This is a contradiction. Similarly, we can show that $\rho_r(X)=r$. We conclude that $|R(X)|=r+1=|C(X)|$. Moreover, $U(XV)=UV$ iff $XV=V$ since $R(XV)=R(V)$ and so $XV=PV$ for some P in S_n. It follows that if $XV\neq V$ then $UXV\neq UV$ and hence $XV=V$ iff $X=T_r$.

Sufficiency: Trivial.

COROLLARY 5. *If A is an element of A_n and $\rho(A)=r$, then there exist $n\times r$ acyclic matrix U and $r\times n$ acyclic matrix V such that $A=UV$. In particular, we can always pick $n\times r$ acyclic matrix U and $r\times n$ acyclic matrix V such that $C(U)=C(A)$ and $R(V)=R(A)$.*

An acyclic matrix A is said to be *bounded* if $a_{ij}=1$ then $a_{hk}=1$ for all $1\leq h\leq i$ and $1\leq k\leq j$.

PROPOSITION 6. *A is an element of A_n iff there exists a bounded matrix B such that $A=PBQ$ for $P,Q\in S_n$.*

3. Green's relations

PROPOSITION 7. *Let A and B be the element of A_n. Then*

i) $A\mathscr{L}B$ *iff* $B_r(A)=B_r(B)$,

ii) $A\mathscr{R}B$ *iff* $B_c(A)=B_c(B)$,

iii) $A\mathscr{D}B$ *iff* $\rho(A)=\rho(B)$.

PROOF. (i) Necessity: Trivial.

Sufficiency: If $B_r(A)=B_r(B)$, then the maximal number of independent rows of A is equal to the maximal number of independent rows of B. Let $C_1, \cdots, C_r$ be the nonzero independent rows of A and B in some order. If we set

$$C=\begin{bmatrix} C_1 \\ \vdots \\ C_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times n}$$

then there exist $X_1, X_2, Y_1, Y_2 \in A_n$ such that $C=X_1A$, $A=X_2C$, $C=Y_1B$, and $B=Y_2C$. It follows that $A\mathscr{L}C$ and $C\mathscr{L}B$ and hence $A\mathscr{L}B$.

(ii) Dual proof holds for $\mathscr{R}$.

(iii) It follows from the fact that $A\mathscr{D}B$ iff there exists $C\in A_n$ such that $B_r(A)=B_r(C)$ and $B_c(C)=B_c(B)$.

Note that if we let $D(r)=\{A\in A_n: \rho(A)=r\}$, then obviously $D(r)$ is a $\mathscr{D}$-class of A_n.

COROLLARY 8. *Each $\mathscr{D}$-class $D(r)$ contains*

$$\begin{bmatrix} T_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}.$$

COROLLARY 9. *Each $\mathscr{L}$ ($\mathscr{R}$)-class of A_n contains exactly one symmetric matrix.*

4. Combinatorial results

Two matrices A and B are equivalent if there exist P, $Q\in S_n$ such that $A=PBQ$.

PROPOSITION 10. *A_n contains $\frac{(2n)!}{n!\,n!}-1$ equivalent classes.*

PROPOSITION 11. $|D(r)|=(r!S(n+1,\ r+1))^2$.

PROOF. Let $n_{\mathscr{L}}(r)$ ($n_{\mathscr{R}}(r)$) be the number of $\mathscr{L}(\mathscr{R})$-classes contained in $D(r)$.

Then by the Inclusion-Exclusion principle,

$$n_{\mathscr{L}}(r)=\sum_{m=r}^{n}(-1)^{m-r}r!\binom{n}{m}(r+1)^{n-m}S(m,r)=n_{\mathscr{R}}(r).$$

Let A be an element of $D(r)$. If H_A denotes the $\mathscr{H}$-class containing A, then $|H_A|=1$. Putting the pieces together we have

$$\begin{aligned}|D(r)| &= |H_A|n_{\mathscr{L}}(r)n_{\mathscr{R}}(r)\\ &=\Big(\sum_{m=r}^{n}(-1)^{m-r}r!\binom{n}{m}(r+1)^{n-m}S(m,r)\Big)^2\\ &=(r!S(n+1,r+1))^2\end{aligned}$$

COROLLARY 12. $|A_n|=\sum_{r=1}^{n}(r!S(n+1,r+1))^2$.

The following proposition is a combination of results proved by the author in [2].

PROPOSITION 13. $|E(D(r))|=\sum_{m=r}^{n}(-1)^{m-r}r!S(m,r)\Big(\frac{r^2+5r+2}{2}\Big)$.

COROLLARY 14. $|E(A_n)|=\sum_{r=1}^{n}\sum_{m=r}^{n}(-1)^{m-r}r!S(m,r)\Big(\frac{r^2+5r+2}{2}\Big)$.

From what has been said we get immediately the following result.

PROPOSITION 15. *A_n is a regular semigroup.*

PROOF. Every $\mathscr{L}(\mathscr{R})$-class contained in A_n contains at least one idempotent by proposition 13. Hence A_n is a regular semigroup by Theorem 2.11 of [3].

Note that A_n is not an inverse semigroup.

Note that we obtained a new regular subsemigroup of B_n, the semigroup of binary relations defined on n elements and therefore we have partially answered B.M. Schein's problem [4].

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REFERENCES

[1] A. Berman and A. Kotzig, *The order of cyclicity of (0,1)-matrices*, Technical Report, No. 315, Centre De Recherches Mathematiques, Universite De Montreal,

Montreal, Canada, August 1973.

[2] K. K. -H. Butler, *Combinatorial properties of binary semigroups*, Periodica Mathematica, Hungarica, 5 (1974), 3—46.

[3] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Survey No. 7, Amer. Math. Soc., Providence, R. I., 1961.

[4] B. M. Schein, *Semigroups of binary relations*, The Proc. of Mini-Conference on Semigroup Theory, Szeged, Hungary, August 1972, p. 22.