Kyungpook Math. J.
Volume 15, Number 1
June, 1975

CATEGORIES IN WHICH EVERY MONO-SOURCE IS INITIAL

By Sung Sa Hong

Dedicated to Prefessor Chung Ki Pahk on his 60th Birthday

1. Introduction

Topologically algebraic functors have been introduced by Y. H. Hong [5], which generalize topological functors introduced by Herrlich [3] and cover most of categories usually studied in algebra, topology and universal topological algebra.

In this paper, it is shown that topologically algebraic functors are precisely those functors with left adjoints such that the domain categories are (epi, initial)-factorizable. Observing that for monadic functors and algebraic functors, every monosource in the domain categories is initial with respect to the functor in question, we will give some sufficient conditions for those functors to be topologically algebraic. Using this and applying results in [5], one can deduce more properties in the domain category of such a functor from those in its codomain category. In particular, we note that topologically algebraic functors detect colimits and limits.

For general categorical background and terminology we refer to [2], except that every subcategory of a category will be assumed to be full and isomorphism closed.

2. Topologically algebraic functors

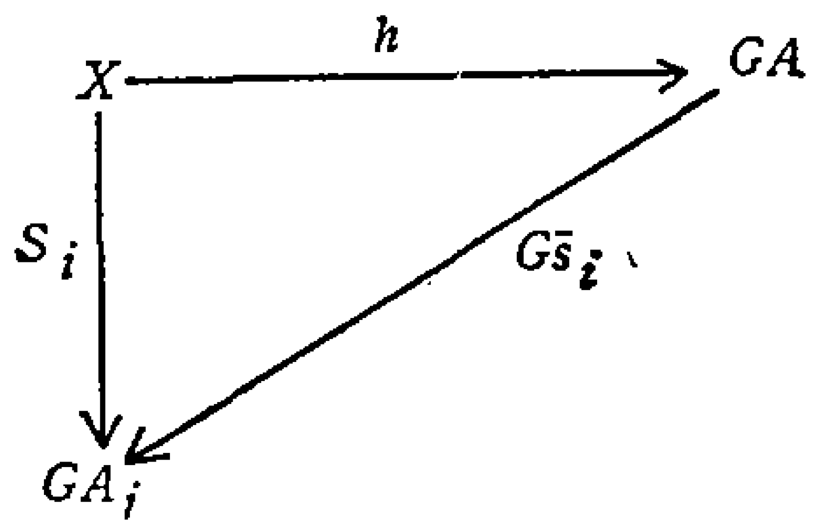
DEFINITION 2.1. Let $G:\mathbf{A}\longrightarrow\mathbf{B}$ be a functor. A source $(A,\ A\xrightarrow{f_i}A_i)_{i\in I}$ in $\mathbf{A}$ is called *G-initial* (or *initial with respect to G*) if for any source $(A',\ A'\xrightarrow{g_i}A_i)_{i\in I}$ in $\mathbf{A}$ and any $\mathbf{B}$-morphism $h:GA'\longrightarrow GA$ with $(Gf_i)h=Gg_i(i\in I)$ there exists a unique A-morphism $\bar{h}:A'\longrightarrow A$ with $G\bar{h}=h$ and $f_i\bar{h}=g_i$ $(i\in I)$.

The category of sets (topological spaces, compact Hausdorff spaces resp.) and maps (continuous maps resp.) will be denoted by **Set** (**Top**, **Comp** resp.).

EXAMPLE. A source $(A,\ (f_i)_I)$ in **Top** is initial with respect to the underlying set functor **Top**$\longrightarrow$**Set** iff A is endowed with the initial topology with respect to $(f_i)_I$. For any subcategory **K** of **Comp**, it is easy to show that every point-separating-source in **K** is initial with respect to its underlying set functor.

DEFINITION 2.2. Let $G: \mathbf{A} \longrightarrow \mathbf{B}$ be a functor. A $\mathbf{B}$-morphism $g: X \longrightarrow GA$ is said to *G-generate* A if for any pair $A \underset{s}{\overset{r}{\rightrightarrows}} B$ of $\mathbf{A}$-morphisms $(Gr)g=(Gs)g$ implies $r=s$.

DEFINITION 2.3. [5]. A functor $G: \mathbf{A} \longrightarrow \mathbf{B}$ is called ***topologically algebraic*** if for each family $(A_i)_{i \in I}$ of $\mathbf{A}$-objects and each source $(X, X \xrightarrow{s_i} GA_i)_{i \in I}$ in B there exists a G-initial source $(A, A \xrightarrow{\bar{s}_i} A_i)_{i \in I}$ in $\mathbf{A}$ and a $\mathbf{B}$-morphism $h: X \longrightarrow GA$ which G-generates A such that the diagram



commutes for each $i \in I$.

It has been shown [5] that every topologically algebraic functor is faithful, has a left adjoint and detects limits and colimits, i.e., if $D: I \longrightarrow A$ is diagram and GD has a limit (colimit, resp.) then so has D.

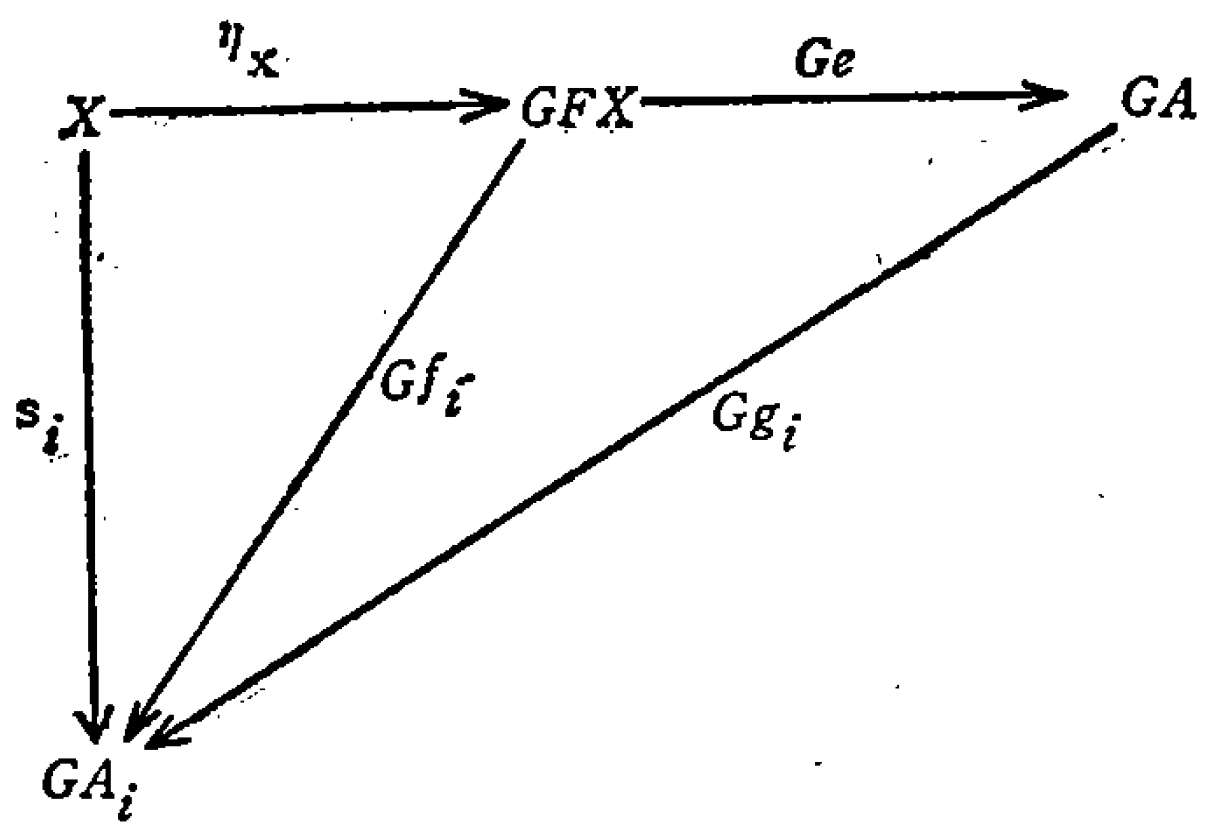
DEFINITION 2.4. For a category $\mathbf{A}$, let $\mathbf{E}$ ($\mathbf{M}$ resp.) be a class of morphisms (sources resp.). Then $\mathbf{A}$ is called *(E, M)-factorizable* if every source $(A, A \xrightarrow{f_i} A_i)_I$ in A has a factorization, $A \xrightarrow{f_i} A_i = A \xrightarrow{e} B \xrightarrow{g_i} A_i (i \in \mathrm{I})$ with $e \in E$ and $(B, (g_i)) \in M$. And $\mathbf{A}$ is called an *(E, M)-category* if it is $(\mathbf{E}, \mathbf{M})$-factorizable and for a morphism $e: A \longrightarrow B$ in $\mathbf{E}$ and a source $(C, C \xrightarrow{m_i} A_i)_I$ in $\mathbf{M}$, and for any morphism $f: A \longrightarrow C$ and any source $(B, B \xrightarrow{f_i} A_i)_I$ with $m_i f = f_i e (i \in I)$, there is a unique morphism $g: B \longrightarrow C$ such that $ge=f$ and $m_i g = f_i (i \in I)$.

THEOREM 2.5. *A functor $G: \mathbf{A} \longrightarrow \mathbf{B}$ is topologically algebraic iff G has a left adjoint and $\mathbf{A}$ is (epi, G-initial)-factorizable.*

PROOF. ($\Rightarrow$). It remains to show that $\mathbf{A}$ is (epi, G-initial)-factorizable. Let $(A, (f_i)_I)$ be a source in $\mathbf{A}$. Then there is a G-initial source $(B, (g_i)_I)$ and a morphism $h: GA \longrightarrow GB$ which G-generates B such that $(Gg_i)h = Gf_i$ $(i \in I)$. Hence there is a unique $\mathbf{A}$-morphism $e: A \longrightarrow B$ with $Ge=h$ and $f_i = g_i e$ $(i \in I)$. $Ge=h$ G-generating B, e is an epimorphism ; $f_i = g_i e (i \in I)$ is the desired factorization.

($\Leftarrow$). Let F be a left adjoint of G. Let $(A_i)_{i \in I}$ be a family of $\mathbf{A}$-objects and

$(X, X\xrightarrow{s_i}GA_i)_{i\in I}$ a source in **B**. Let $\eta_X : X\longrightarrow GFX$ be a front adjunction of X. Then there is a unique $f_i : FX\longrightarrow A_i$ with $(Gf_i)\eta_X=s_i$ $(i\in I)$. Let $FX\xrightarrow{f_i}A_i=FX\xrightarrow{e}A\xrightarrow{g_i}A_i$ be an (epi, G-initial)-factorization. Hence we have the following commutative diagram



Since e is an epimorphism and η_X G-generates FX, $(Ge)\eta_X$ G-generates A. This completes the proof.

REMARK. In the above theorem, G is also topologically algebraic if G has a left adjoint and **A** is (**E**, G-initial)-factorizable for any class **E** of epimorphisms in **A** (see the last argument of the proof).

We will denote the category of universal algebras of fixed type τ and homomorphisms by A(τ). For a subcategory **T** of **Top** and a subcategory **A** of A(τ), the category of universal topological algebras whose underlying algebras belong to **A** and underlying spaces belong to **T**, and continuous homomorphisms will be denoted by **TA**. It is shown [5] that for an epireflective subcategory **T** of **Top** and an epireflective subcategory **A** of **A**(τ), the underlying set functor of **TA** is topologically algebraic. We note that **Comp** is not epireflective in **Top**.

COROLLARY 2.6. *Let* **K** *be either the category* **Comp** *or the category* **ZComp** *of zero-dimensional compact Hausdorff spaces and continuous maps. For an epireflective subcategory* A *of* A(τ) (*equivalently* A *is closed under products and subalgebras*), *the underlying set* (*space, algebra resp.*) *functor* G : KA⟶Set (KA⟶K, KA⟶A *resp.*) *is topologically algebraic.*

PROOF. By the adjoint functor theorem (with the "solution set" condition), one can easily conclude that the functors have left adjoints. Since every mono-source in K or A is initial with respect to its underlying set functor, it is obvious that every mono-source in **KA** is also G-initial. Furthermore. **KA** is an (epi, monosources) category; **KA** is (epi, G-initial)-factorizable. This completes the proof.

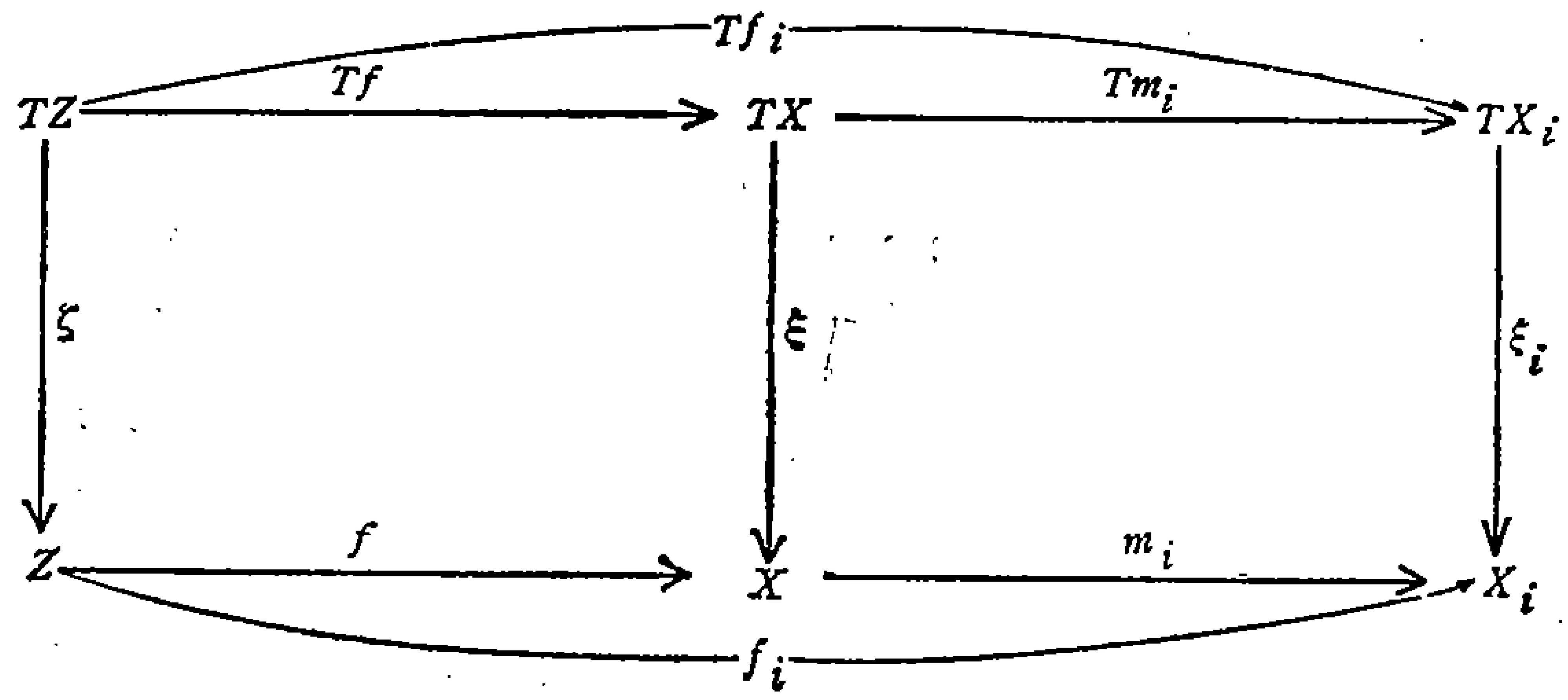
3. Monadic functors and algebraic functors

The reader is referred to [1] and [6] for monadic functors, except that monadic functors in this paper are defined in a way for the comparison functors to be equivalences of categories instead of isomorphisms as in [6], and to [2] for algebraic functors, i.e., functors which have left adjoints and preserve and reflect regular epimorphisms.

We note again that every mono-source in a subcategory of $\mathbf{A}(\tau)$ is initial with respect to the underlying set functor.

PROPOSITION 3.1. If a functor $G:\mathbf{A}\longrightarrow\mathbf{B}$ is either monadic or algebraic, then every mono-source in $\mathbf{A}$ is G-initial.

PROOF. Suppose G is monadic and F is a left adjoint of G. Let $(T=GF,\ \eta,\ \mu)$ be the monad in $\mathbf{B}$ defined by the adjunction, i.e., $\eta:1_{\mathbf{B}}\longrightarrow T=GF$ is the front adjunction and $\mu=G\varepsilon F:T^2\longrightarrow T$ while $\varepsilon:FG\longrightarrow 1_{\mathbf{A}}$ is the back adjunction. The category of T-algebras (see [1] or VI.2, [6]) will be denoted by $\mathbf{B}^T$: objects of $\mathbf{B}^T$ are the pairs (X,ξ) where X is a $\mathbf{B}$-object and $\xi:TX\longrightarrow X$ is a $\mathbf{B}$-morphism satisfying $\xi(T\xi)=\xi\mu_X$ and $\xi\eta_X=1_X$ and a morphism $f:(X,\xi)\longrightarrow(X',\xi')$ in $\mathbf{B}^T$ is given by a $\mathbf{B}$-morphism $f:X\longrightarrow X'$ with $f\xi=\xi'(Tf)$. The comparison functor $K:\mathbf{A}\longrightarrow\mathbf{B}^T$ (see [1] or Theorem 1, VI.3, [6]) such that $KF=F^T$ and $G^TK=G$, where G^T is the underlying object functor of $\mathbf{B}^T$ into $\mathbf{B}$ and F^T is its left adjoint. Since K is an equivalence of categories, K preserves mono-sources, and moreover a source $(A,\ (f_i)_I)$ in $\mathbf{A}$ is G-initial iff the source $(KA,\ (Kf_i)_I)$ is G^T-initial. Hence it is enough to show that every mono-source in $\mathbf{B}^T$ is G^T-initial. Let $((X,\xi),\ (X,\xi)\xrightarrow{m_i}(X_i,\xi_i))_{i\in I}$ be a mono-source in $\mathbf{B}^T$. For any source $((Z,\zeta),\ (Z,\zeta)$



$\xrightarrow{f_i}(X_i, \xi_i))_I$ in $\mathbf{B}^T$ and any $\mathbf{B}$-morphism $f : G^T(Z,\zeta) \longrightarrow G^T(X,\xi)$ with $G^T f_i = (G^T m_i)f$, i.e., $f_i = m_i f (i \in I)$ in $\mathbf{B}$, we have the preceeding diagram, in which the outer rectangle and the right square commute for each $i \in I$ due to the definition of morphisms of $\mathbf{B}^T$. Since G^T preserves mono-sources, $(X, (m_i))$ is a mono-source in $\mathbf{B}$; $\xi(Tf) = f\zeta$. Hence $f : (Z,\zeta) \longrightarrow (X,\xi)$ is actually a morphism of $\mathbf{B}^T$ and $G^T f = f$. For the case of G being algebraic, the proof is essentially contained in Proposition 32.7 and 32.8 [2].

REMARK. 1) The above proposition might give an easy criterion for a functor to be not monadic. For instance, underlying set functors of subcategories of **Top** except those of subcategories of **Comp** need not be monadic, for every mono-source in those categories need not be initial.

2) If $G : \mathbf{A} \longrightarrow \mathbf{B}$ is either monadic or algebraic and $\mathbf{B}$ is well powered then so is $\mathbf{A}$.

In what follows, $\mathbf{E}$ ($\mathbf{M}$ resp.) denotes a class of epimorphisms (mono-sources resp.) in a category in question.

Using Proposition 3.1 and Remark 1) of Theorem 2.5, the following is immediate.

THEOREM 3.2. *Let a functor* $G : \mathbf{A} \longrightarrow \mathbf{B}$ *be either monadic or algebraic. Then* G *is topologically algebraic if* $\mathbf{A}$ *is* $(\mathbf{E}, \mathbf{M})$*-factorizable. In case,* G *detects colimits and limits.*

PROPOSITION 3.3. *Let* $G : \mathbf{A} \longrightarrow \mathbf{B}$ *be a monadic functor with a left adjoint* F. *If* $\mathbf{B}$ *is an* $(\mathbf{E}, \mathbf{M})$*-category and* $\mathbf{E}$ *is closed under* $T = GF$, *then* G *is topologically algebraic.*

PROOF. Let $\mathbf{B}^T$ be the category of T-algebras explained in the proof of Proposition 3.1. Since the comparison functor $K : \mathbf{A} \longrightarrow \mathbf{B}^T$ is an equivalence, and equivalences of categories detect (epi, mono-sources)-factorizations, it is enough to show that $\mathbf{B}^T$ is (epi, mono-sources)-factorizable. For a source $((X,\xi), (X,\xi) \xrightarrow{f_i} (X_i, \xi_i))_{i \in I}$ in $\mathbf{B}^T$, let $X \xrightarrow{f_i} X_i = X \xrightarrow{e} Z \xrightarrow{m_i} X_i (i \in I)$ be the $(\mathbf{E}, \mathbf{M})$-factorization in $\mathbf{B}$. Since $Te \in E$ and $(Z, (m_i)) \in \mathbf{M}$, there is a unique morphism $\zeta : TZ \longrightarrow Z$ in $\mathbf{B}$ such that the diagram

$$\begin{array}{ccccc} TX & \xrightarrow{Te} & TZ & \xrightarrow{Tm_i} & TX_i \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \xi_i} \\ X & \xrightarrow{e} & Z & \xrightarrow{m_i} & X_i \end{array}$$

commutes for each $i \in I$.

It is well known that $(Z, \zeta) \in \mathbf{B}^T$ follows from the fact that $(X, \xi) \in \mathbf{B}^T$ and $e\xi = \zeta(Te)$, and the fact e, Te, and T^2e are epimorphisms. Moreover, e and m_i are actually morphisms in $\mathbf{B}^T$. Hence $(X, \xi) \xrightarrow{f_i} (X_i, \xi_i) = (X, \xi) \xrightarrow{e} (Z, \zeta) \xrightarrow{m_i} (X_i, \xi_i)$ is the desired factorization, for G^T reflects epimorphisms and mono-sources.

REMARK. With the same conditions in the above proposition, one can easily conclude that $\mathbf{B}^T$ is actually an $(\mathbf{E}^T, \mathbf{M}^T)$-category, where $\mathbf{E}^T = (G^T)^{-1}$ $(\mathbf{E})$ and $\mathbf{M}^T = (G^T)^{-1}(\mathbf{M})$.

Since **Set** is an (epi, mono-sources)-category and epimorphisms in **Set** are precisely retractions, we have the following from the above proposition (also see Theorem 4.2 [4] and the following Proposition 3.6).

COROLLARY 3.4. *Every monadic functor* **Set** *is topogically algebraic.*

Since every left adjoint functor preserves regular epimorphisms, the following is immediate from Proposition 3.3, while we refer to [4] for regular categories and regular functors.

COROLLARY 3.5. *Let* $\mathbf{A} \longrightarrow \mathbf{B}$ *be a monadic functor. If G preserves regular epimorphisms and* **B** *is a regular category, then G is topologically algebraic.*

PROPOSITION 3.6. *A regular functor* $G : \mathbf{A} \longrightarrow \mathbf{B}$ *is topologically algebraic.*

PROOF. By Proposition 2.3 [4], G is algebraic and **A** is a regular so that G is topologically algebraic due to Theorem 3.2.

Carleton University
Ottawa, Canada

REFERENCES

[1] S. Eilenberg and J. C. Moore, *Adjoint functors and triples,* Illinois J. Math. 9 (1965), 381—398.

[2] H. Herrlich and G. E. Strecker, *Category Theory,* Allyn and Bacon. Boston (1973).

[3] H. Herrlich, *Topological functors.* Gen. Topology Appl. 4 (1974), 125—142.

[4] H. Herrlich, *Regular categories and regular functors,* Canad. J. Math. 26 (1974). 709—720.

[5] Y. H. Hong, *Studies on categories of universal topological algebras,* Doct. Dissertation, McMaster University (1974).

[6] S. MacLane, *Categories for the working mathematician.* Springer-Verlag, New York (1971).