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THE H-FUNCTION OF TWO VARIABLES

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1. Introduction

The double Mellin-Barnes type contour integral appearing in this paper will be referred to as the H-function of two variables throughout our present study and will be defined and represented in the following manner [14, p. 117] :

$$H\left[\begin{matrix}\begin{pmatrix}0, & n_1\\ p_1, & q_1\end{pmatrix}\\ \begin{pmatrix}m_2, & n_2\\ p_2, & q_2\end{pmatrix}\\ \begin{pmatrix}m_3, & n_3\\ p_3, & q_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(a_j;\alpha_j, A_j)_{1,p_1}\\ (b_j;\beta_j, B_j)_{1,q_1}\\ (c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{1,q_2}\\ (e_j, E_j)_{1,p_3}\\ (f_j, F_j)_{1,q_3}\end{matrix}\right|\begin{matrix}x\\ \\ y\\ \\ \end{matrix}\right]=(1/2\pi i)^2\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)x^s y^t ds\, dt \tag{1.1}$$

where

$$\phi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\ \prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)},$$

$$\theta_1(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\ \prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\ \prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j s)},$$

$$\theta_2(t)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)\ \prod_{j=1}^{n_3}\Gamma(1-e_j+E_j t)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j t)\ \prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j t)}.$$

Also in (1.1) x and y are not equal to zero and an empty product is interpreted as unity. The nonnegative integers n_i, p_i, q_i $(i=1,2,3)$ and m_2, m_3 are such that $0\leq n_i\leq p_i$, $q_1\geq 0$, $0\leq m_j\leq q_j$ $(i=1,2,3,\ ; j=2,3)$ and all the Greek letters $\alpha,\beta,\gamma,\delta$, and capital letters A,B,E,F are assumed to be positive quantities.

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j s)(j=1,\cdots,m_2)$ lie to the right and those of $\Gamma(1-a_j+\alpha_j s+A_j t)$ $(j=1, \cdots, n_1)$, $\Gamma(1-c_j+\gamma_j s)(j=1, \cdots, n_2)$ lie to the left of the contour.

The contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j t)$ $(j=1, \cdots, m_3)$, lie to the right and those of $\Gamma(1-a_j+\alpha_j s+A_j t)(j=1, \cdots, n_1)$, $\Gamma(1-e_j+E_j t)$ $(j=1, \cdots, n_3)$ lie to the left of the contour.

NOTATIONS USED. In (1.1) above and throughout the paper when there is no possibility of being misunderstood, we shall write:

$(a_j; \alpha_j, A_j)_{1,p}$ for $(a_1; \alpha_1, A_1), \cdots, (a_p; \alpha_p, A_p)$:

$(a_j; \alpha_j, 1)_{1,p}$ for $(a_1; \alpha_1, 1), \cdots, (a_p; \alpha_p, 1)$;

$(a_j; 1, 1)_{1,p}$ for $(a_1; 1, 1), \cdots, (a_p; 1, 1)$;

$(a_j, \alpha_j)_{1,p}$ for $(a_1, \alpha_1), \cdots, (a_p, \alpha_p)$:

$(a_j, \alpha_j)_{n+1,p}$ for $(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$;

$(a_j, 1)_{1,p}$ for $(a_1, 1), \cdots, (a_p, 1)$:

(a_p) for $a_1, \cdots, a_p$;

$\left(\Delta(N, a_j) : \frac{\alpha_j}{N}, \frac{A_j}{N}\right)_{1,p}$ for $\left(\Delta(N, a_1) : \frac{\alpha_1}{N}, \frac{A_1}{N}\right), \cdots, \left(\Delta(N, a_p) : \frac{\alpha_p}{N}, \frac{A_p}{N}\right)$:

$\left(\Delta(N, a) : \frac{\alpha}{N}, \frac{A}{N}\right)$ for $\left(\frac{a}{N} : \frac{\alpha}{N}, \frac{A}{N}\right), \cdots, \left(\frac{a+N-1}{N} ; \frac{\alpha}{N}, \frac{A}{N}\right)$:

$\left(\Delta(N, a_j), \frac{\alpha_j}{N}\right)_{1,p}$ for $\left(\Delta(N, a_1), \frac{\alpha_1}{N}\right), \cdots, \left(\Delta(N, a_p), \frac{\alpha_p}{N}\right)$:

$\left(\Delta(N, a), \frac{\alpha}{N}\right)$ for $\left(\frac{a}{N}, \frac{\alpha}{N}\right), \left(\frac{a+1}{N}, \frac{\alpha}{N}\right), \cdots, \left(\frac{a+N-1}{N}, \frac{\alpha}{N}\right)$:

and $H[x, y]$ for the H-function of two variables defined in (1.1).

Also to save space, we shall write the H-function of two variables defined by (1.1) in the following manner;

$$H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, n_1 \\ p_1, q_1 \end{pmatrix} & \begin{array}{l}(a_j; \alpha_j, A_j)_{1,p_1} \\ (b_j; \beta_j, B_j)_{1,q_1}\end{array} & x \\ \cdots & \cdots & \\ \cdots & \cdots & y \end{array}\right]$$

when there is a change only in n_1, p_1, q_1, a_j, α_j, A_j $(j=1, \cdots, p_1)$ b_j, β_j, B_j $(j=1, \cdots, q_1)$ while all its other parameters remain unchanged.

Following the lines of Braaksma [5, p. 278], it can be shown that the function defined by (1.1) is an analytic function of x and y, if

(i) $R=\sum_1^{p_1}(\alpha_j)+\sum_1^{p_2}(\gamma_j)-\sum_1^{q_1}(\beta_j)-\sum_1^{q_2}(\delta_j)<0,$

(ii) $S=\sum_1^{p_1}(A_j)+\sum_1^{p_3}(E_j)-\sum_1^{q_1}(B_j)-\sum_1^{q_3}(F_j)<0.$

From the study of a known result [7, p. 50, (6)], we see that the integral converges, if

(iii) $u=\sum_1^{n_1}(\alpha_j)-\sum_{n_1+1}^{p_1}(\alpha_j)-\sum_1^{q_1}(\beta_j)+\sum_1^{m_2}(\delta_j)-\sum_{m_2+1}^{q_2}(\delta_j)+\sum_1^{n_2}(\gamma_j)-\sum_{n_2+1}^{p_2}(\gamma_j)>0$

(iv) $v=\sum_1^{n_1}(A_j)-\sum_{n_1+1}^{p_1}(A_j)-\sum_1^{q_1}(B_j)+\sum_1^{m_3}(F_j)-\sum_{m_3+1}^{q_3}(F_j)+\sum_1^{n_3}(E_j)-\sum_{n_3+1}^{p_3}(E_j)>0$

(v) $|\arg x|<(1/2)u\pi$

(vi) $|\arg y|<(1/2)v\pi.$

Series representation of the H-function of two variables

If (i) $\delta_m(d_j+\lambda)\neq\delta_j(d_m+\mu)$, for $j\neq m$: $j, m=1, \cdots, m_2$: $\lambda, \mu=0, 1, 2, \cdots$

(ii) $F_n(f_k+\rho)\neq F_k(f_n+\nu)$, for $k\neq n$: $k, n=1, \cdots, m_3$: $\rho, \nu=0, 1, 2, \cdots$

(iii) $x\neq0$, $y\neq0$, $R<0$, $S<0$ (where R, S stand for the quantities as given above),

then all the poles of (1.1) are simple and the value of the integral (1.1) can be evaluated as sum of the residues. Proceeding on the lines similar to that of Braaksma [5, p. 278], we obtain the following interesting and new result:

$$H[x,y]=\sum_{m=1}^{m_2}\sum_{n=1}^{m_3}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\frac{d_m+\mu}{\delta_m}+A_j\frac{f_n+\nu}{F_n}\right)\prod_{\substack{j=1\\j\neq m}}^{m_2}\Gamma\left(d_j-\delta_j\frac{d_m+\mu}{\delta_m}\right)}{\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\frac{d_m+\mu}{\delta_m}+B_j\frac{f_n+\nu}{F_n}\right)}$$

$$\cdot\frac{\prod_{j=1}^{n_2}\Gamma\left(1-c_j+\gamma_j\frac{d_m+\mu}{\delta_m}\right)\prod_{\substack{j=1\\j\neq n}}^{m_3}\Gamma\left(f_j-F_j\frac{f_n+\nu}{F_n}\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\frac{d_m+\mu}{\delta_m}-A_j\frac{f_n+\nu}{F_n}\right)\prod_{j=m_2+1}^{q_2}\Gamma\left(1-d_j+\delta_j\frac{d_m+\mu}{\delta_m}\right)}$$

$$\times\frac{\prod_{j=1}^{n_3}\Gamma\left(1-e_j+E_j\frac{f_n+\nu}{F_n}\right)(-1)^{\mu+\nu}(x)^{\frac{d_m+\mu}{\delta_m}}}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\frac{d_m+\mu}{\delta_m}\right)}$$

$$\times \frac{(y)^{\frac{f_n+\nu}{F_n}}}{\prod_{j=m_3+1}^{q_3} \Gamma\left(1-f_j+F_j \frac{f_n+\nu}{F_n}\right) \prod_{j=n_3+1}^{p_3} \Gamma\left(e_j-E_j \frac{f_n+\nu}{F_n}\right) \delta_m F_n \mu! \nu!}. \tag{1.2}$$

From (1.2), it follows that

$$H[x,y] = 0(|x|^{\alpha}, |y|^{\beta}) \text{ for small values of } x \text{ and } y, \tag{1.3}$$

where $R<0$, $S<0$ (R, S stand for the quantities as mentioned above), $\alpha = \min \operatorname{Re}\left(\frac{d_j}{\delta_j}\right)$ $(j=1, \cdots, m_2)$ and $\beta = \min \operatorname{Re}\left(\frac{f_j}{F_j}\right)$ $(j=1, \cdots, m_3)$.

By considering the behaviour of the Gamma functions involved in $H[x,y]$ defined by (1.1), it can be shown that for $n_1=0$,

$$H[x,y] = 0(|x|^{\rho}, |y|^{\sigma}) \text{ for large values of } x \text{ and } y \tag{1.4}$$

where $\rho = \max \operatorname{Re}\left(\frac{c_j-1}{\gamma_j}\right)$ $(j=1, \cdots, n_2)$, $\sigma = \max \operatorname{Re}\left(\frac{e_j-1}{E_j}\right)$ $(j=1, \cdots, n_3)$ and the conditions (i) to (vi) given above are satisfied.

2. Properties of the H-function of two variables

The obvious changes of variables in the integral (1.1) give us the following three relations:

$$(a)\quad x^{\rho}y^{\sigma}H[x,y] = H\left[\begin{matrix} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} (a_j+\rho\alpha_j+\sigma A_j;\ \alpha_j,\ A_j)_{1,p_1} \\ (b_j+\rho\beta_j+\sigma B_j;\ \beta_j,\ B_j)_{1,q_1} \\ (c_j+\rho\gamma_j,\ \gamma_j)_{1,p_2} \\ (d_j+\rho\delta_j,\ \delta_j)_{1,q_2} \\ (e_j+\sigma E_j,\ E_j)_{1,p_3} \\ (f_j+\sigma E_j,\ F_j)_{1,q_3} \end{matrix} \right| \begin{matrix} x \\ \\ y \end{matrix}\right] \tag{2.1}$$

$$(b)\quad H[x,y] = cd\ H\left[\begin{matrix} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} (a_j;\ c\alpha_j,\ dA_j)_{1,p_1} \\ (b_j;\ c\beta_j,\ dB_j)_{1,q_1} \\ (c_j,\ c\gamma_j)_{1,p_2} \\ (d_j,\ c\delta_j)_{1,q_2} \\ (e_j,\ dE_j)_{1,p_3} \\ (f_j,\ dF_j)_{1,q_3} \end{matrix} \right| \begin{matrix} x \\ \\ y \end{matrix}\right] \tag{2.2}$$

where $c>0$, $d>0$.

(c) $$H[x,y]=(2\pi)^{(1-N)\left(\sum_{i=1}^{3}\left(n_i-\frac{p_i+q_i}{2}\right)+\sum_{i=2}^{3}(m_i)\right)}N^{\mu}$$

$$\times H\left[\begin{array}{c|c|c} \begin{pmatrix}0, & Nn\\ Np_1, & Nq_1\end{pmatrix} & \begin{array}{l}\left(\Delta(N,a_j):\frac{\alpha_j}{N},\frac{A_j}{N}\right)_{1,p_1}\\ \left(\Delta(N,b_j):\frac{\beta_j}{N},\frac{B_j}{N}\right)_{1,q_1}\end{array} & xN^{-\rho}\\ \begin{pmatrix}Nm_2, & Nn_2\\ Np_2, & Np_q\end{pmatrix} & \begin{array}{l}\left(\Delta(N.\ c_j),\frac{\gamma_j}{N}\right)_{1,p_2}\\ \left(\Delta(N,d_j),\frac{\delta_j}{N}\right)_{1,q_2}\end{array} & \\ \begin{pmatrix}Nm_3, & Nn_3\\ Np_3, & Nq_3\end{pmatrix} & \begin{array}{l}\left(\Delta(N,e_j),\frac{E_j}{N}\right)_{1,p_3}\\ \left(\Delta(N,f_j),\frac{F_j}{N}\right)_{1,q_3}\end{array} & yN^{-\sigma}\end{array}\right] \tag{2.3}$$

where (i) $\mu=\sum_1^3\frac{p_i-q_i}{2}+\sum_1^{q_1}(b_j)+\sum_1^{q_2}(d_j)+\sum_1^{q_3}(f_j)-\sum_1^{p_1}(a_j)-\sum_1^{p_2}(c_j)-\sum_1^{p_3}(e_j)$

(ii) $\rho=\sum_1^{q_1}(\beta_j)+\sum_1^{q_2}(\delta_j)-\sum_1^{p_1}(\alpha_j)-\sum_1^{p_2}(\gamma_j)$

(iii) $\sigma=\sum_1^{q_1}(B_j)+\sum_1^{q_3}(F_j)-\sum_1^{p_1}(A_j)-\sum_1^{p_3}(E_j)$

(iv) N is a positive integer ≥ 2.

The various symbols used here are explained in Sec. 1.*

3. Special cases of (1.1)

(i) If we take $\alpha_i=A_i$, $\beta_j=B_j$ $(i=1,\cdots,p_1;\ j=1,\cdots,q_1)$ in (1.1), it reduces to the functions which in essence are similar to those studied by Bora and Kalla [4], Pathak [15], Chaturvedi [6] and Mathur [13].

(ii) If we take all α's, β's, γ's, δ's, A's, B's, E's and F's equal to unity

$$H\left[\begin{array}{c|l|c} \begin{pmatrix}0, & n_1\\ p_1, & q_1\end{pmatrix} & \begin{array}{l}(a_j;1,1)_{1,p_1}\\ (b_j;1,1)_{1,q_1}\end{array} & x\\ \begin{pmatrix}m_2, & n_2\\ p_2, & q_2\end{pmatrix} & \begin{array}{l}(c_j,1)_{1,p_2}\\ (d_j,1)_{1,q_2}\end{array} & \\ \begin{pmatrix}m_3, & n_3\\ p_3, & q_3\end{pmatrix} & \begin{array}{l}(e_j,1)_{1,p_3}\\ (f_j,1)_{1,q_3}\end{array} & y\end{array}\right]=G\left[\begin{array}{c|c|c} \begin{pmatrix}0, & n_1\\ p_1, & q_1\end{pmatrix} & \begin{array}{l}(a_{p_1})\\ (b_{q_1})\end{array} & x\\ \begin{pmatrix}m_2, & n_2\\ p_2, & q_2\end{pmatrix} & \begin{array}{l}(c_{p_2})\\ (d_{q_2})\end{array} & \\ \begin{pmatrix}m_3, & n_3\\ p_3, & q_3\end{pmatrix} & \begin{array}{l}(e_{p_3})\\ (f_{q_3})\end{array} & y\end{array}\right]=G[x,y] \tag{3.1}$$

* The result (2.3) can be proved, if we apply the Gamma multiplication formula [7, p. 4] in the definition of the H-function of two variables given by (1.1).

in (1.1), we get the function (3.1) which in essence is similar to the functions studied earlier by a number of persons notably Agarwal [1], Sharma [16].

From the definition of the H-function of two variables, it is evident that

$$G[x,y]=(1/2i)^2\int_{L_1}\int_{L_2}\phi(s+t)\ \theta'(s,t)\ x^s\ y^t\ ds\ dt \tag{3.2}$$

where

$$\phi(s+t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+s+t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s-t)\prod_{j=1}^{q_1}\Gamma(1-b_j+s+t)},$$

$$\theta'(s,t)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-s)\prod_{j=1}^{n_2}\Gamma(1-c_j+s)\prod_{j=1}^{m_3}\Gamma(f_j-t)\prod_{j=1}^{n_3}\Gamma(1-e_j+t)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-s)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+t)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-t)}.$$

[The nature of the contours L_1 and L_2, the conditions of convergence of the double integral (3.2), various conditions satisfied by the parameters and a number of other properties of $G[x,y]$ function follow directly from those of the H-function of two variables, if we take all Greek letters $\alpha, \beta, \gamma, \delta$ and Capital letters A, B, E, F equal to unity in it.]

(iii) If we put $m_2=m_3=1$, $n_i=p_i$ $(i=1,2,3)$, $\delta_1=F_1=1$, $d_1=f_1=0$, replace q_2 by q_2+1, q_3 by q_3+1, a_j by $1-a_j$ $(j=1, \cdots, p_1)$, b_j by $1-b_j$ $(j=1, \cdots, q_1)$, c_j by $1-c_j$ $(j=1, \cdots, p_2)$, d_{j+1} by $1-d_j$ $(j=1, \cdots, q_2)$, e_j by $1-e_j$ $(j=1, \cdots, p_3)$, f_{j+1} by $1-f_j$ $(j=1, \cdots, q_3)$ in (1.1) and make use of (1.2) in it, then the H-function of two variables reduces to the Generalised Kampé de Fériet function introduced by Srivastva [18, p. 199]:

$$H\left[\begin{matrix}\begin{pmatrix}0, & p_1\\ p_1, & q_1\end{pmatrix}\\ \begin{pmatrix}1, & p_2\\ p_2, & q_2+1\end{pmatrix}\\ \begin{pmatrix}1, & p_3\\ p_3, & q_3+1\end{pmatrix}\end{matrix}\left|\begin{matrix}(1-a_j;\alpha_j, A_j)_{1,p_1}\\ (1-b_j;\beta_j, B_j)_{1,q_1}\\ (1-c_j, \gamma_j)_{1,p_2}\\ (0,1), (1-d_j, \delta_j)_{1,q_2}\\ (1-e_j, E_j)_{1,p_3}\\ (0,1), (1-f_j, F_j)_{1,q_3}\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right]=S[x,y]$$

$$=S^{p_1:p_2:p_3}_{q_1:q_2:q_3}\left[\begin{matrix}(a_j;\alpha_j, A_j)_{1,p_1}:(c_j, \gamma_j)_{1,p_2}:(e_j, E_j)_{1,p_3}:x,\ y\\ (b_j;\beta_j, B_j)_{1,q_1}:(d_j, \delta_j)_{1,q_2}:(f_j, F_j)_{1,q_3}\end{matrix}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p_1} \Gamma(a_j+\alpha_j m+A_j n) \prod_{j=1}^{p_2} \Gamma(c_j+\gamma_j m) \prod_{j=1}^{p_3} \Gamma(e_j+E_j n) x^m y^n}{\prod_{j=1}^{q_1} \Gamma(b_j+\beta_j m+B_j n) \prod_{j=1}^{q_2} \Gamma(d_j+\delta_j m) \prod_{j=1}^{q_3} \Gamma(f_j+F_j n)\; m!\,n!} \tag{3.3}$$

where

$$1+\sum_1^{q_1}(\beta_j)+\sum_1^{q_2}(\delta_j)-\sum_1^{p_1}(\alpha_j)-\sum_1^{p_2}(\gamma_j)>0$$

$$1+\sum_1^{q_1}(B_j)+\sum_1^{q_3}(F_j)-\sum_1^{p_1}(A_j)-\sum_1^{p_3}(E_j)>0.$$

The function given by (3.3) is also quite general in nature and reduces to the Kampé de Fériet function [2], if we take $p_2=p_3$, $q_2=q_3$ and all α's, β's, γ's, δ's, A's, B's, E's and F's equal to one in it, which is itself a generalisation of the well known Appell functions [7, p. 224—26].

(iv) If we take $n_1=p_1=0$ in (1.1), it degenerates in to the product of two of two H-functions of one variable and we have the following relations which has also been given by Mittal and Gupta [14, p. 119] :

$$H\left[\begin{array}{c|c|c} \binom{0,0}{0,0} & \begin{array}{c}\cdots\cdots\\ \cdots\cdots\end{array} & \\ \binom{m_2,\ n_2}{p_2,\ q_2} & \begin{array}{c}(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array} & x \\ \binom{m_3,\ n_3}{p_3,\ q_3} & \begin{array}{c}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array} & y \end{array}\right] = H^{m_2,n_2}_{p_2,q_2}\left[x\,\middle|\,\begin{array}{c}(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array}\right] H^{m_3,n_3}_{p_3,q_3}\left[y\,\middle|\,\begin{array}{c}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array}\right] \tag{3.4}$$

(v) Also, if we take $m_3=1$, $n_3=p_3$, $f_1=0$, replace q_3 by q_3+1 in (1.1) put all A's, B's, E's, F's equal to unity in it and let $y\to 0$ therein, we get the following relation by virtue of (1.1) and known results [9, p. 598, (4.1) : 7, p. 208, (5)] :

$$\underset{y\to 0}{\mathrm{Lt}}\, H\left[\begin{array}{c|c|c} \binom{0\ ,n_1}{p_1,q_1} & \begin{array}{c}(a_j;\alpha_j,1)_{1,p_1}\\ (b_j;\beta_j,1)_{1,q_1}\end{array} & x \\ \binom{m_2,n_2}{p_2,\ q_2} & \begin{array}{c}(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array} & \\ \binom{1\ ,p_3}{p_3,\ q_3+1} & \begin{array}{c}(e_j,1)_{1,p_3}\\ (0,1),(f_j,1)_{1,q_3}\end{array} & y \end{array}\right]$$

$$= \frac{\prod_{j=1}^{p_3} \Gamma(1-e_j)}{\prod_1^{q_3} \Gamma(1-f_j)}\; H^{m_2,n_1+n_2}_{p_1+p_2,q_1+q_2}\left[x\,\middle|\,\begin{array}{c}(a_j,\alpha_j)_{1,n_1},\ (c_j,\gamma_j)_{1,p_2},\ (a_j,\alpha_j)_{n_1+1,p_1}\\ (d_j,\delta_j)_{1,q_2},\ (b_j,\beta_j)_{1,q_1}\end{array}\right] \tag{3.5}$$

where $p_1+p_3<q_1+q_3+1$.

(vi) It we set $m_3=1$, $f_1=f_0$, $F_1=F_0$, replace q_3 by q_3+1, f_{j+1} by f_j and F_{j+1} by F_j $(j=1, \cdots, q_3)$ in (1. 1) and compare the results (1. 2) and [5, p. 278] we get the following interesting result:

$$H\left[\begin{matrix}\binom{0,\ n_1}{p_1,\ q_1}\\ \binom{m_2,\ n_2}{p_2,\ q_2}\\ \binom{1,\ n_3}{p_3,\ q_3+1}\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}\\(b_j;\beta_j,B_j)_{1,q_1}\\(c_j,\gamma_j)_{1,p_2}\\(d_j,\delta_j)_{1,q_2}\\(e_j,E_j)_{1,p_3}\\(f_0,F_0),(f_j,F_j)_{1,q_3}\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right]$$

$$=\sum_{w=0}^{\infty}\frac{(-1)^w\,(y)^{\rho_w}\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\rho_w)}{F_0 w!\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\rho_w)\prod_{j=1}^{q_3}\Gamma(1-f_j+F_j\rho_w)}$$

$$\times H^{m_2,n_1+n_2}_{p_1+p_2,q_1+q_2}\left[x\left|\begin{matrix}(c_j,\gamma_j)_{1,n_2},\ (a_j-A_j\rho_w,\alpha_j)_{1,p_1},\ (c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,q_2},\ (b_j-B_j\rho_w,\beta_j)_{1,q_1}\end{matrix}\right.\right] \tag{3.6}$$

where $\rho_w=\dfrac{f_0+w}{F_0}$.

(vii) Again, if we put $n_1=m_3=1$, $p_1=q_3=2$, $n_3=p_3=q_1=0$, $\alpha_1=\alpha_2=\sigma$, $A_1=A_2=F_1=F_2=1$, $a_1=a$, $a_2=b$, $f_1=f$, $f_2=h$ and $y=1$ in (1, 1) and make use of known results [9, p. 594, 598 ; 7, p. 208, (5), p. 104, (46)] in it, then we get:

$$H\left[\begin{matrix}\binom{0,\ 1}{2,\ 0}\\ \binom{m_2,\ n_2}{p_2,\ q_2}\\ \binom{1,\ 0}{0,\ 2}\end{matrix}\left|\begin{matrix}(a;\sigma,1),\ (b;\sigma,1)\\ \cdots\cdots\cdots\\(c_j,\gamma_j)_{1,p_2}\\(d_j,\delta_j)_{1,q_2}\\ \cdots\cdots\cdots\\(f,1),(h,1)\end{matrix}\right|\begin{matrix}x\\ \\ 1\end{matrix}\right]$$

$$=H^{m_2+1,n_2+1}_{p_2+4,q_2+1}\left[x\left|\begin{matrix}(a-f,\sigma),\ (c_j,\gamma_j)_{1,p_2},\ (b-f,\sigma),\ (b-h,\sigma),(a-h,\sigma)\\(a+b-f-h-1,2\sigma),\ (d_j,\delta_j)_{1,q_2}\end{matrix}\right.\right] \tag{3.7}$$

where $\mathrm{Re}(a+b-f-h-1)>0$

(vii) Also, if we put $n_1=p_1=m_3=q_3=2$, $q_1=n_3=p_3=0$, $a_1=a$, $a_2=b$, $f_1=0$, $f_2=c$, $\alpha_1=\rho$, $\alpha_2=\sigma$, $A_1=A_2=F_1=F_2=1$ in (1. 1) and use the known results [9, p. 594, 598] and the following result [17]

$$G_{2,2}^{2,2}\left[y\,\middle|\,\begin{matrix}a, b\\0, c\end{matrix}\right]=\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(1-a+c)\Gamma(1-b+c)}{\Gamma(2-a-b+c)}$$
$$\times{}_2F_1(1-a, 1-b;2-a-b+c;1-y) \tag{3.8}$$

in it, then we get the following useful and interesting result:

$$H\left[\begin{matrix}\begin{pmatrix}0, & 2\\2, & 0\end{pmatrix}\\ \begin{pmatrix}m_2, & n_2\\p_2, & q_2\end{pmatrix}\\ \begin{pmatrix}2, & 0\\0, & 2\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}(a;\rho, 1),\ (b;\sigma, 1)\\ \cdots\cdots\cdots\\ (c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{1,q_2}\\ \cdots\cdots\cdots\\ (0, 1),\ (c, 1)\end{matrix}\,\middle|\,\begin{matrix}x\\ \\ y\end{matrix}\right]$$
$$=\sum_{r=0}^{\infty}\frac{(1-y)^r}{r!}H_{p_2+4, q_2+1}^{m_2, n_2+4}\left[x\,\middle|\,\begin{matrix}(a-r, \rho), (b-r, \sigma), (a-c, \rho), (b-c, \sigma), (c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{1,q_2}, (a+b-c-1-r, , \rho+\sigma)\end{matrix}\right] \tag{3.9}$$

(ix) If we let $y\rightarrow 1$, in (3.9), then we get

$$H\left[\begin{matrix}\begin{pmatrix}0\ , & 2\\2\ , & 0\end{pmatrix}\\ \begin{pmatrix}m_2, & n_2\\p_2, & q2\end{pmatrix}\\ \begin{pmatrix}2\ , & 0\\0\ , & 2\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}(a;\rho, 1),\ (b;\sigma, 1)\\ \cdots\cdots\cdots\\ (c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{1,q_2}\\ \cdots\cdots\cdots\\ (0, 1),\ (c, 1)\end{matrix}\,\middle|\,\begin{matrix}x\\ \\ y\end{matrix}\right]$$
$$=H_{p_2+4, q_2+1}^{m_2, n_2+4}\left[x\,\middle|\,\begin{matrix}(a, \rho), (b, \sigma), (a-c, \rho), (b-c, \sigma), (c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{1,q_2}, (a+b-c-1, \rho+\sigma)\end{matrix}\right]$$

4. In this section, we shall give the following Multiplication formula for the H-function of two variables which can be proved easily by making use of Gamma multiplication formula [7, p. 4] in the definition of this function given by (1.1):

$$H\left[\begin{matrix}\begin{pmatrix}0, & n_1\\n_1+p_1, & q_1\end{pmatrix}\\ \begin{pmatrix}m_2, & n_2\\n_2+p_2, & m_2+q_2\end{pmatrix}\\ \begin{pmatrix}m_3, & n_3\\n_3+p_3, & m_3+q_3\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}(a_j;\alpha_j, A_j)_{1,n_1}, (a'_j;\alpha'_j, A'_j)_{1,p_1}\\ (b_j;\beta_j, B_j)_{1,q_1}\\ (c_j, \gamma_j)_{1,p_2}, (c'_j, \gamma'_j)_{1,p_2}\\ (d_j, \delta_j)_{1,m_2}, (d'_j, \delta'_j)_{1,q_2}\\ (e_j, E_j)_{1,n_3}, (e'_j, E'_j)_{1,p_3}\\ (f_j, F_j)_{1,m_3}, (f'_j, F'_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}x\\ \\ y\end{matrix}\right]$$
$$=(2\pi)^{\frac{1}{2}\left\{\sum_{i=1}^{3}(n_i-p_i-q_i-N_i+P_i+Q_i)+\sum_{i=2}^{3}(m_i-M_i)\right\}}\prod_{1}^{n_1}(N_j)^{\frac{1}{2}-a_j}\prod_{1}^{p_1}(P_j)^{\frac{1}{2}-a'_j}$$

$$\times \prod_1^{q_1}(Q_j)^{b_j-\frac{1}{2}} \prod_1^{n_2}(M_j)^{\frac{1}{2}-c_j} \prod_1^{m_2}(L_j)^{d_j-\frac{1}{2}} \prod_1^{p_2}(R_j)^{\frac{1}{2}-c'_j} \prod_1^{q_2}(S_j)^{d'_j-\frac{1}{2}}$$

$$\times \prod_1^{n_3}(T_j)^{\frac{1}{2}-e_j} \prod_1^{m_3}(U_j)^{f_j-\frac{1}{2}} \prod_1^{p_3}(V_j)^{\frac{1}{2}-e'_j} \prod_1^{q_3}(W_j)^{f'_j-\frac{1}{2}}$$

$$\times H\left[\begin{matrix} \begin{pmatrix} 0\,, & N_1 \\ N_1+P_1, & Q_1 \end{pmatrix} \\ \\ \begin{pmatrix} M_2, & N_2 \\ N_2+P_2, & M_2+Q_2 \end{pmatrix} \\ \\ \begin{pmatrix} M_3, & N_3 \\ N_3+P_3, & M_3+Q_3 \end{pmatrix} \end{matrix} \left| \begin{matrix} \left(\Delta(N_j,a_j);\frac{\alpha_j}{N_j},\frac{A_j}{N_j}\right)_{1,n_1}, \left(\Delta(P_j,a'_j);\frac{\alpha'_j}{P_j},\frac{A'_j}{P_j}\right)_{1,p_1} \\ \left(\Delta(Q_j,b_j);\frac{\beta_j}{Q_j},\frac{B_j}{Q_j}\right)_{1,q_1} \\ \left(\Delta(M_j,c_j),\frac{\gamma_j}{M_j}\right)_{1,n_2}, \left(\Delta(R_j,c'_j),\frac{\gamma'_j}{R_j}\right)_{1,p_2} \\ \left(\Delta(L_j,d_j),\frac{\delta_j}{L_j}\right)_{1,m_2}, \left(\Delta(s_j,d'_j),\frac{\delta'_j}{S_j}\right)_{1,q_2} \\ \left(\Delta(T_j,e_j),\frac{E_j}{T_j}\right)_{1,n_3}, \left(\Delta(V_j,e'_j),\frac{E'_j}{V_j}\right)_{1,p_3} \\ \left(\Delta(U_j,f_j),\frac{F_j}{U_j}\right)_{1,m_3}, \left(\Delta(W_j,f'_j),\frac{F'}{W_j}\right)_{1,q_3} \end{matrix} \right| \begin{matrix} \frac{\alpha\beta\delta\varepsilon}{\gamma\mu\nu}x \\ \\ \frac{\alpha'\beta'\delta'\varepsilon'}{\gamma'\mu'\nu'}y \end{matrix}\right] \tag{4.1}$$

where (i) $N_1=\sum_1^{n_1}(N_j)$, $N_2=\sum_1^{n_2}(M_j)$, $N_3=\sum_1^{n_3}(T_j)$, $M_2=\sum_1^{m_2}(L_j)$, $M_3=\sum_1^{m_3}(U_j)$,

$P_1=\sum_1^{p_1}(P_j)$, $P_2=\sum_1^{p_2}(R_j)$, $P_3=\sum_1^{p_3}(V_j)$, $Q_1=\sum_1^{q_1}(Q_j)$, $Q_2=\sum_1^{q_2}(S_j)$,

$Q_3=\sum_1^{q_3}(W_j)$.

(ii) All capital letters M's, M's, P's, Q's are positive integers.

(iii) $\alpha=\prod_1^{n_1}(N_j)^{\alpha_j}$, $\beta=\prod_1^{p_1}(P_j)^{\alpha'_j}$, $\gamma=\prod_1^{q_1}(Q_j)^{\beta_j}$, $\delta=\prod_1^{n_2}(M_j)^{\gamma_j}$, $\varepsilon=\prod_1^{p_2}(R_j)^{\gamma'_j}$

$\mu=\prod_1^{m_2}(L_j)^{\delta_j}$, $\nu=\prod_1^{q_2}(S_j)^{\delta'_j}$, $\alpha'=\prod_1^{n_1}(N_j)^{A_j}$, $\beta'=\prod_1^{p_1}(P_j)^{A'_j}$, $\gamma'=\prod_1^{q_1}(Q_j)^{B_j}$,

$\delta'=\prod_1^{n_3}(T_j)^{E_j}$, $\varepsilon'=\prod_1^{p_3}(V_j)^{E'_j}$, $\mu'=\prod_1^{m_3}(U_j)^{F_j}$, $\nu'=\prod_1^{q_3}(W_j)^{F'_j}$,

(iv) $\left(\Delta(N_j,a_j);\ \frac{\alpha_j}{N_j},\ \frac{A_j}{N_j}\right)_{1,n}$ stand for the n pairs $\left(\Delta(N_1,a_1);\ \frac{\alpha_1}{N_1},\ \frac{A_1}{N_1}\right)$,

$\cdots$, $\left(\Delta(N_n,a_n);\ \frac{\alpha_n}{N_n},\ \frac{A_n}{N_n}\right)$;

(v) $\left(\Delta(N_j,a_j),\ \frac{\alpha_j}{N_j}\right)_{1,n}$ stand for the parameters $\left(\Delta(N_1,a_1),\ \frac{\alpha_1}{N_1}\right)$, $\cdots$,

$\left(\Delta(N_n, a_n), \dfrac{\alpha_n}{N_n}\right)$, and other symbols are explained in section 1.

Special cases of (4.1) : If we take $\alpha_j=A_j=N_j$ $(j=1,\cdots,n_1)$, $\alpha'_j=A'_j=P_j(j=1,\cdots,p_1)$, $\beta_j=B_j=Q_j(j=1,\cdots,q_1)$, $\gamma_j=M_j(j=1,\cdots,n_2)$, $\gamma'_j=R_j$ $(j=1,\cdots,p_2)$, $\delta_j=L_j(j=1,\cdots,m_2)$, $\delta'_j=S_j(j=1,\cdots,q_2)$, $E_j=T_j(j=1,\cdots,n_3)$, $E'_j=V_j(j=1,\cdots,p_3)$, $F_j=U_j$ $(j=1,\cdots,m_3)$, $F'_j=W_j(j=1,\cdots,q_3)$ in (4.1), then we get the following relation between the H-function of two variables and the G-function of two variables defined in (3.2):

$$H\left[\begin{matrix}\begin{pmatrix}0, & n_1\\ n_1+p_1, & q_1\end{pmatrix}\\ \begin{pmatrix}m_2, & n_2\\ n_2+p_2, & m_2+q_2\end{pmatrix}\\ \begin{pmatrix}m_3, & n_3\\ n_3+p_3, & m_3+q_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,\alpha_j)_{1,n_1}, (a'_j,\alpha'_j,\alpha'_j)_{1,p_1}\\ (b_j;\beta_j,\beta_j)_{1,q_1}\\ (c_j,\gamma_j)_{1,n_2}, (c'_j,\gamma'_j)_{1,p_2}\\ (d_j,\delta_j)_{1,m_2}, (d'_j,\delta'_j)_{1,q_2}\\ (e_j,E_j)_{1,n_3}, (e'_j,E'_j)_{1,p_3}\\ (f_j,F_j)_{1,m_3}, (f'_j,F'_j)_{1,q_3}\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right]$$

$$=(2\pi)^{\frac{1}{2}\left\{\sum_{i=1}^{3}(n_i-p_i-q_i-N_i+P_i+Q_i)+\sum_{i=2}^{3}(m_i-M_i)\right\}}\prod_1^{n_1}(N_j)^{\frac{1}{2}-a_j}\prod_1^{p_1}(P_j)^{\frac{1}{2}-a'_j}$$

$$\times\prod_1^{q_1}(Q_j)^{b_j-\frac{1}{2}}\prod_1^{n_2}(M_j)^{\frac{1}{2}-c_j}\prod_1^{m_2}(L_j)^{d_j-\frac{1}{2}}\prod_1^{p_2}(R_j)^{\frac{1}{2}-c'_j}\prod_1^{q_2}(S_j)^{d'_j-\frac{1}{2}}$$

$$\times\prod_1^{n_3}(T_j)^{\frac{1}{2}-e_j}\prod_1^{m_3}(U_j)^{f_j-\frac{1}{2}}\prod_1^{p_3}(V_j)^{\frac{1}{2}-e'_j}\prod_1^{q_3}(W_j)^{f'_j-\frac{1}{2}}$$

$$\times G\left[\begin{matrix}\begin{pmatrix}0, & N_1\\ N_1+P_1, & Q_1\end{pmatrix}\\ \begin{pmatrix}M_2, & N_2\\ N_2+P_2, & M_2+Q_2\end{pmatrix}\\ \begin{pmatrix}M_3, & N_3\\ N_3+P_3, & M_3+Q_3\end{pmatrix}\end{matrix}\left|\begin{matrix}\{\Delta(N_{n_1},a_{n_1})\}, \{\Delta(P_{p_1},a'_{p_1})\}\\ \{\Delta(Q_{q_1},b_{q_1})\}\\ \{\Delta(M_{n_2},c_{n_2})\}, \{\Delta(R_{p_2},c'_{p_2})\}\\ \{\Delta(L_{m_2},d_{m_2})\}, \{\Delta(S_{q_2},d'_{q_2})\}\\ \{\Delta(T_{n_3},e_{n_3})\}, \{\Delta(V_{p_3},e'_{p_3})\}\\ \{\Delta(U_{m_3},f_{m_3})\}, \{\Delta(W_{q_3},f'_{q_3})\}\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right] \quad (4.2)$$

where (i) $\{\Delta(M_{m_1}, b_{m_1})\}$ is used to denote the sequence of parameters $\Delta(M_1,b_1)$, $\Delta(M_2,b_2)$, $\cdots$, $\Delta(M_{m_1},b_{m_1})$

(ii) $\Delta(n,a)$ stands for $\frac{a}{n}$, $\frac{a+1}{n}$, $\cdots$, $\frac{a+n-1}{n}$ and the other symbols stand for the same quantities as mentioned in (4.1).

If we reduce the H-function of two variables occuring in (4.1) to the H-function of one variable with the help of equation (3.5) and adjust the parameters in the H-function of one variable in a suitable manner, we get a result earlier obtained by Gupta and Jain [11, p. 26].

5. In this section, we shall obtain the following two formulae involving the rth derivative of the H-function of two variables. The conditions of validity are $\rho>0$, $\sigma>0$, $R<0$, $S<0$, $u>0$, $v>0$, $|\arg y|<(1/2)u\pi$, $|\arg z|<(1/2)v\pi$, $\mathrm{Re}\left(\lambda+\rho\frac{d_i}{\delta_i}+\sigma\frac{f_j}{F_j}+1\right)>0$ ($i=1, \cdots, m_2$, $j=1, \cdots, m_3$) and the rth derivative of $x^{\lambda}H[yx^{\rho}, zx^{\sigma}]$ should exist (R, S, u, v are the quantities as mentioned in section 1).

$$\text{(i)}\quad \frac{d^r}{dx^r}\{x^{\lambda}H[yx^{\rho}, zx^{\sigma}]\}=x^{\lambda-r}\ H\left[\begin{array}{c|c|c}\begin{pmatrix}0, & n_1+1\\ p_1+1, & q_1+1\end{pmatrix} & \begin{array}{l}(-\lambda;\rho,\sigma), (a_j;\alpha_j, A_j)_{1,p_1}\\ (b_j;\beta_j, B_j)_{1,q_1}, (r-\lambda;\rho,\sigma)\end{array} & yx^{\rho}\\ \cdots & \cdots & \\ \cdots & \cdots & zx^{\sigma}\end{array}\right] \tag{5.1}$$

$$\text{(ii)}\quad \left(x\frac{d}{dx}\right)^r\{x^{\lambda}H[yx^{\rho}, zx^{\sigma}]\}=x^{\lambda}\ H\left[\begin{array}{c|c|c}\begin{pmatrix}0, & n_1+r\\ p_1+r, & q_1+r\end{pmatrix} & \begin{array}{l}(-\lambda;\rho,\sigma)_r, (a_j;\alpha_j, A_j)_{1,p_1}\\ (b_j;\beta_j, B_j)_{1,q_1}, (1-\lambda;\rho,\sigma)_r\end{array} & yx^{\rho}\\ \cdots & \cdots & zx^{\sigma}\\ \cdots & \cdots & \end{array}\right] \tag{5.2}$$

In (5.2) the symbol $(a,\sigma)_r$ is used to denote (a,σ), (a,σ), $\cdots$, r times.

PROOF of (5.1): We have [8, p. 129, 130]:

$$L\left\{x^m\frac{d^n}{dx^n}[f(x)]\,;s\right\}=\left(-\frac{d}{ds}\right)^m[s^nL\{f(x);s\}] \tag{5.3}$$

$$L\{x^n f(x);s\}=(-1)^n\frac{d^n}{dx^n}[L\{f(x);s\}] \tag{5.4}$$

where

$$L\{f(x);s\}=\int_0^{\infty}e^{-sx}f(x)\,dx. \tag{5.5}$$

Now, on taking $f(x)=x^{\lambda}H[yx^{\rho}, zx^{\sigma}]$ in the right hand side of (5.3), then by virtue of the following result [14, p. 22]:

$$L\{x^{\lambda}H[yx^{\rho}, zx^{\sigma}]; s\}$$

$$= s^{-\lambda-1}H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, n_1+1 \\ p_1+1, q_1 \end{pmatrix} & \begin{array}{l} (-\lambda;\rho,\sigma), (a_j;\alpha_j, A_j)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1} \end{array} & ys^{-\rho} \\ \cdots & \cdots & \\ \cdots & \cdots & zs^{-\sigma} \end{array}\right] \tag{5.6}$$

its value reduced to the following expression:

$$\left(-\frac{d}{ds}\right)^R\left\{s^{-r-\lambda-1}H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, n_1+1 \\ p_1+1, q_1 \end{pmatrix} & \begin{array}{l} (-\lambda;\rho,\sigma), (a_j;\alpha_j, A_j)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1} \end{array} & ys^{-\rho} \\ \cdots & \cdots & \\ \cdots & \cdots & zs^{-\sigma} \end{array}\right]\right\} \tag{A}$$

Taking the inverse Laplace transform of the quantity within the crooked bracket in (A), with the help of the following result [14, p. 122]:

$$s^{-\lambda-1}H[ys^{-\rho}, zs^{-\sigma}] = L\{x^{\lambda}H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1+1 \end{pmatrix} & \begin{array}{l} (a_j;\alpha_j, A_j,)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1}, (-\lambda;\rho,\sigma) \end{array} & yx^{\rho} \\ \cdots & \cdots & \\ \cdots & \cdots & zx^{\sigma} \end{array}\right]; s\} \tag{5.6}$$

and then applying (5.4) in the result thus obtained, the right hand side of (5.3) becomes equal to

$$L\{x^{\lambda-r+R}H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1+1 \\ p_1+1, & q_1+1 \end{pmatrix} & \begin{array}{l} (-\lambda;\rho,\sigma), (a_j;\alpha_j, A_j)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1}, (r-\lambda;\rho,\sigma) \end{array} & yx^{\rho} \\ \cdots & \cdots & \\ \cdots & \cdots & zx^{\sigma} \end{array}\right]; s\} \tag{B}$$

Therefore (5.3) is equivalent to

$$L\left\{x^R\frac{d^r}{dx^r}[x^{\lambda}H[yx^{\rho}, zx^{\sigma}]]; s\right\}$$

$$= L\{x^{\lambda-r+R}H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1+1 \\ p_1+1, & q_1+1 \end{pmatrix} & \begin{array}{l} (-\lambda;\rho,\sigma), (a_j;\alpha_j, A_j)_{1,p_1} \\ (b_j;\beta_j, B_j)_{1,q_1}, (r-\lambda;\rho,\sigma) \end{array} & yx^{\rho} \\ \cdots & \cdots & \\ \cdots & \cdots & zx^{\sigma} \end{array}\right]; s\} \tag{5.7}$$

Interpreting (5.7) with the help of Lerch's theorem [12] which states that every function has got a unique image in the Laplace transform, we arrive at the result (5.1).

The result (5.2) can easily proved by successive application of (5.1),

If we take all A's, B's, E's, F's equal to unity, $f_1=0$ and let $z\to 0$ in (5.1) and (5.2), we get the results obtained earlier by Gupta and Jain [10, p. 190, 191] by virtue of (3.5), which in turn reduce formulae involving the rth derivative of Meijer's G-function given earlier by Bhise [3, p. 350—51].

On account of the general nature of the H-function of two variables the (5.1) and (5.2) can be used as key formulae for obtaining the rth derivative of the various important special functions involving one variable and two variables as indicated in section 3, but we shall do not record them due to lack of space.

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