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REDUCTION FORMULAS FOR GENERALIZED LAURICELLA FUNCTIONS*

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1. Introduction

Making use of the familiar Pochhammer symbol $(\alpha)_n$, where

$$(1.1)\qquad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\begin{cases}1, & \text{if } n=0,\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n=1,\ 2,\ 3,\ \cdots,\end{cases}$$

the generalized Lauricella function, introduced by Srivastava and Daoust [5, p. 454], is defined by

$$(1.2)\quad F^{A:B':\cdots:B^{(r)}}_{C:D':\cdots:D^{(r)}}\begin{pmatrix}z_1\\ \vdots\\ z_r\end{pmatrix}\equiv F^{A:B':\cdots:B^{(r)}}_{C:D':\cdots:D^{(r)}}\left(\begin{matrix}[(a):\theta',\ \cdots,\ \theta^{(r)}]:\\ [(c):\phi',\ \cdots,\ \phi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\ \cdots,\ z_r\right)$$

$$=\sum_{m_1,\cdots,m_r=0}^{\infty}\Omega(m_1,\ \cdots,\ m_r)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!},$$

where, for convenience,

$$(1.3)\quad \Omega(m_1,\cdots,m_r)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j'+\cdots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b_j')_{m_1\phi_j'}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j'+\cdots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}},$$

the coefficients

$$(1.4)\qquad \begin{cases}\theta_j^{(i)},\ j=1,\ \cdots,\ A, & \phi_j^{(i)},\ j=1,\ \cdots,\ B^{(i)},\\ \psi_j^{(i)},\ j=1,\ \cdots,\ C, & \delta_j^{(i)},\ j=1,\ \cdots,\ D^{(i)},\ 1\le i\le r,\end{cases}$$

are real and positive, and (a) is taken to abbreviate the sequence of A parameters $a_1,\ \cdots,\ a_A$, $(b^{(i)})$ abbreviates the squence of $B^{(i)}$ parameters

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$$b_1^{(i)},\ \cdots,\ b_{B^{(i)}}^{(i)},$$

with similar interpretations for (c), $(d^{(i)})$, etc., $i=1, \cdots, r$. Also, for the sake of brevity, we use the following notations throughout this paper:

$$((a))_n = \prod_{j=1}^{A} (a_j)_n, \tag{1.5}$$

$$((b^{(i)}))_n = \prod_{j=1}^{B^{(i)}} (b_j^{(i)})_n, \quad i=1, \cdots, r, \text{ etc.} \tag{1.6}$$

Indeed it is easy to observe that, when $r=1$ and $r=2$, (1.2) would correspond respectively to the generalized hypergeometric function, introduced by Wright ([8] and [9]), and to the generalization in two variables of Kampé de Fériet's double hypergeometric function, introduced by Srivastava and Daoust (cf., e.g., [5], p. 450).

The multiple hypergeometric series in (1.2) is absolutely convergent if

$$\Delta_i \equiv 1 + \sum_{j=1}^{C} \phi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \geq 0, \quad (i=1, \cdots, r), \tag{1.7}$$

wherein the equality holds when $|z_i| < \rho_i$, $i=1, \cdots, r$, with the ρ_i defined by equation (5.3), p. 157 in [6], For a detailed discussion of these convergence conditions, one may see § 5 of reference [6].

The aforementioned convergence conditions will be assumed to hold throughout the present work. Also, the following two well-known summation theorems will be required in our proofs (cf., e.g., [1], p. 99 (3) and p. 103. (2)).

$${}_2F_1(-n,\ a;\ c;\ 1) = \frac{(c-a)_n}{(c)_n}. \tag{1.8}$$

$${}_3F_2(-n,\ a,\ b;\ c,\ d;\ 1) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \tag{1.9}$$

provided that $c+d=a+b-n+1$, n being a nonnegative integer.

The formulas to be established here are

$$\mathrm{F}^{A:B';\cdots;B^{(j-1)};B^{(j)}+1;B^{(j+1)};\cdots;B^{(r)}}_{C:D';\cdots;D^{(j-1)};D^{(j)}+1;D^{(j+1)};\cdots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(j-1)},1,\theta^{(j+1)},\cdots,\theta^{(r)}]: \\ [(c):\phi',\cdots,\phi^{(j-1)},1,\phi^{(j+1)},\cdots,\phi^{(r)}]:\end{matrix}\right.$$
$$\begin{matrix}[(b'):\phi'];\cdots;[(b^{(j-1)}):\phi^{(j-1)}];[(b^{(j)}):1],\ [\delta+m:1];[(b^{(j+1)}):\phi^{(j+1)}]; \\ [(d'):\delta'];\cdots;[(d^{(j-1)}):\delta^{(j-1)}];[(d^{(j)}):1],\ [\delta:1];[(d^{(j+1)}):\delta^{(j+1)}];\end{matrix}$$
$$\left.\begin{matrix}\cdots;[(d^{(r)}):\phi^{(r)}]; \\ \cdots;[(d^{(r)}):\delta^{(r)};] \end{matrix}; z_1,\cdots,z_r\right) \tag{1.10}$$

$$=\sum_{n=0}^{m}\binom{m}{n}\frac{((a))_n((b^{(j)}))_n}{(\delta)_n((c))_n((d^{(j)}))_n}z_j^n F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\left(\begin{matrix}[(a)+n:\theta',\cdots,\theta^{(j-1)},1,\theta^{(j+1)},\cdots,\theta^{(r)}]:\\ [(c)+n:\phi',\cdots,\phi^{(j-1)},1,\phi^{(j+1)},\cdots,\phi^{(r)}]:\end{matrix}\right.$$
$$\begin{matrix}[(b'):\phi'];\cdots;[(b^{(j-1)}):\phi^{(j-1)}];[(b^{(j)})+n:1];\\ [(d'):\delta'];\cdots;[(d^{(j-1)}):\delta^{(j-1)}];[(d^{(j)})+n:1];\end{matrix}$$
$$\left.\begin{matrix}[(b^{(j+1)}):\phi^{(j+1)}];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d^{(j+1)}):\delta^{(j+1)}];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r,\right)$$

where m is a positive integer and $1\leq j\leq r$, and

(1.11) $$F^{A:B'+2:B''+2:B^{(3)};\cdots;B^{(r)}}_{C+1:D':\quad\cdots\quad;D^{(r)}}\left(\begin{matrix}[(a):1,1,\theta^{(3)},\cdots,\theta^{(r)}]:\\ [\alpha+\beta:1,1],[(c):1,1,\phi^{(3)},\cdots,\phi^{(r)}]:\end{matrix}\right.$$
$$\begin{matrix}[\alpha:1],[\beta:1],[(b)':1];[\alpha:1],[\beta:1];[(b''):1];\\ [(d'):1];\qquad\qquad [(d''):1];\end{matrix}$$
$$\left.\begin{matrix}[(b^{(3)}):\phi^{(3)}];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(d^{(3)}):\delta^{(3)}];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n((a))_{2n}((b'))_n((b''))_n}{n!(\alpha+\beta)_n((c))_{2n}((d'))_n((d''))_n}(-z_1z_2)^n F^{A+2:B';\cdots;B^{(r)}}_{C+1:D';\cdots;D^{(r)}}\left(\begin{matrix}[\alpha+n:1,1],[\beta+n:1,1],\\ [\alpha+\beta+n:1,1],\end{matrix}\right.$$
$$\begin{matrix}[(a)+2n:1,1,\theta^{(3)},\cdots,\theta^{(r)}]:[(b')+n:1];[(b'')+n:1];[(b^{(3)}):\phi^{(3)}];\\ [(c)+2n:1,1,\phi^{(3)},\cdots,\phi^{(r)}]:[(d')+n:1];[d'')+n:1];[(d^{(3)}):\delta^{(3)}];\end{matrix}$$
$$\left.\begin{matrix}\cdots;[(b^{(r)}):\phi^{(r)}];\\ \cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\cdots,z_r\right),$$

where a parameter like $[\alpha+\beta:1,1]$ corresponds to the factor $(\alpha+\beta)_{m_1+m_2}$ in the denominator of $\Omega(m_1,\cdots,m_r)$ given by (1.3). Similarly for numerator parameters.

2. Derivations of formulas (1.10) and (1.11)

In order to prove (1.10), we start from its right-hand member Δ, say, replace the generalized Lauricella function by its multiple power series given by (1.2), and then make use of the elementary identities

(2.1) $$(\alpha)_n(\alpha+n)_m=(\alpha)_{m+n},\quad \binom{m}{n}=\frac{(-1)^n(-m)_n}{n!},\quad 0\leq n\leq m,$$

which are immediate consequences of the definition (1.1).

We thus find that

(2.2) $$\Delta=\sum_{m_1,\cdots,m_r=0}^{\infty}\Omega'(m_1,\cdots,m_r)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}\sum_{n=0}^{\min(m,m_j)}\frac{(-m)_n(-m_j)_n}{(\delta)_n\ n!},$$

where $\Omega'(m_1, \cdots, m_r)$ is given by (1.3) with $\theta_h^{(j)}=1$, $h=1, \cdots, A$; $\phi_i^{(j)}=1$, $i=1, \cdots, B^{(j)}$; $\psi_k^{(j)}=1$, $k=1, \cdots, C$; and $\delta_l^{(j)}=1$, $l=1, \cdots, D^{(j)}$; $1\leq j\leq r$.

From (2.1), the left-hand side of formula (1.10) would follow at once if we sum the innermost series, which is ${}_2F_1(-m, -m_j; \delta; 1)$, by using the Vandermonde theorem (1.8).

Our proof of the other formula (1.11) is similar; use is made here of Saalschütz's theorem (1.9) instead of the summation formula (1.8).

3. Particular cases

First of all we note that for $r=1$, our formula (1.10) would yield a recent result involving generalized hypergeometric functions given by Srivastava [7] which, in turn, reduces to an earlier formula of Rösler [3].

In the special case when $r=2$, if the positive constants θ's, ϕ's, ψ's and δ's are all equated to 1, formula (1.10) would reduce to the following interesting results for the (modified) Kampé de Fériet function of two variables.

$$(3.1)\quad \sum_{n=0}^{m}\binom{m}{n}\frac{((a))_n((b))_n}{(\delta)_n((c))_n((d))_n}\,x^n\,F^{A:B:B'}_{C:D:D'}\left[\begin{matrix}(a)+n: (b)+n; (b');\\ (c)+n: (d)+n; (d');\end{matrix}\; x,y\right]$$
$$=F^{A:B+1:B'}_{C:D+1:D'}\left[\begin{matrix}(a): \delta+m, (b); (b');\\ (c): \quad \delta, (d); (d');\end{matrix}\; x,y\right].$$

$$(3.2)\quad \sum_{n=0}^{m}\binom{m}{n}\frac{((a))_n((b'))_n}{(\delta)_n((c))_n((d'))_n}\,y^n\,F^{A:B:B'}_{C:D:D'}\left[\begin{matrix}(a)+n: (b): (b')+n;\\ (c)+n: (d); (d')+n;\end{matrix}\; x,y\right]$$
$$=F^{A:B:B'+1}_{C:D:D'+1}\left[\begin{matrix}(a): (b); \delta+m, (b');\\ (c): (d); \quad \delta, \quad (d');\end{matrix}\; x,y\right].$$

A similar special case of formula (1.11) is the summation theorem

$$(3.3)\quad \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n((a))_{2n}((b))_n((b'))_n(-xy)^n}{(\alpha+\beta)_n((c))_{2n}((d))_n((d'))_n\, n!}$$
$$\times F^{A+2:B:B'}_{C+1:D:D'}\left[\begin{matrix}\alpha+n, \beta+n, (a)+2n: (b)+n; (b')+n;\\ \alpha+\beta+n, (c)+2n: (d)+n; (d')+n;\end{matrix}\; x,y\right]$$
$$=F^{A:B+2:B'+2}_{C+1:\ D:\ D'}\left[\begin{matrix}(a): \alpha, \beta, (b); \alpha, \beta, (b');\\ \alpha+\beta, (c): \quad (d); \quad (d');\end{matrix}\; x,y\right].$$

Formula (3.1) when $y=0$, and formula (3.2) when $x=0$, would evidently lead us again to the aforementioned formula of Srivastava [7]. On the other hand, formula (3.3) with $B=B'$ and $D=D'$ is essentially the same as the recent result (11), p.227 of Ragab [2] involving the (ordinary) Kampé de Fériet function.

Incidentally, as an immediate consequence of Vandermonde's theorem(1.8), we can give the following variation of formula (3.3).

$$(3.4)\quad \sum_{n=0}^{\infty} \frac{((a))_{2n}((b))_n((b'))_n(xy)^n}{(\delta)_n((c))_{2n}((d))_n((d'))_n n!} F^{A:B:B'}_{C:D:D'}\left[\begin{matrix}(a)+2n: (b)+n; (b')+n; \\ (c)+2n: (b')+n; (d')+n;\end{matrix}\ x, y\right]$$

$$= F^{A+1:B:B'}_{C:D+1:D'+1}\left[\begin{matrix}\delta, (a): & (b); & (b'); \\ (c): & \delta, (d); & \delta, (d');\end{matrix}\ x, y\right].$$

Finally, we remark that the special cases of (1.10) and (1.11) when $r=3$ would lead us to reduction formulas for certain hypergeometric functions of three variables. More general results, involving the triple hypergeometric function $F^{(3)}[x, y, z]$ defined by Srivastava [4, p. 428], can however, be derived directly by using the method of the preceding section. In view of the similarity of the analysis to be applied, we omit the details of the results thus obtained.

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