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ON r-POTENTS IN A FULL TRANSFORMATION SEMIGROUP

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Dedicated to Prof. Chung Ki Pahk on his 60th Birthday

Let S be a semigroup. Let Z_+ be the set of all positive integers. We define an r-potent in S.

DEFINITION. A non-identity element π in S is called an *r-potent* if $\pi^r=\pi$ and $\pi^i\neq\pi$ for $1<i<r$, where r, $i\in Z_+$. A 2-potent is an idempotent. The main purpose of this note is to prove the following theorem.

THEOREM. *Let r be a fixed positive integer with $2\leq r\leq n$. Let $n>2$. Let T_X be the full transformation semigroup on a set X of n elements. Let S_n be the subsemigroup of T_X consisting of all elements of T_X of rank less than n. Then every element of S_n can be expressed as a finite product of r-potents in S_n.*

REMARK 1. J.M.Howie [2] proved that S_n is a semigroup generated by the idempotents. Our theorem generalizes Howie's theorem [2].

PROOF. We assume that $2<r\leq n$, because of [2]. The proof consists of several steps.

(1) Let $X=\{x_1, x_2, \cdots, x_n\}$. Let $\pi\in S_n$ defined by

$$\pi(x_i)=\begin{cases} x_{i+1} & \text{if } 1\leq i\leq r-2 \\ x_1 & \text{if } r-1\leq i\leq r \\ x_i & \text{if } r+1\leq i\leq n. \end{cases}$$

It is easily seen that π is an r-potent of rank $n-1$ and that π^{r-1} is an idempotent. We set $f=\pi^{r-1}$.

(2) Let D_i $(0\leq i\leq n)$ denote the D-class of T_X consisting of all elements of rank i. Then, by [1, Theorem 2.20], for any idempotent e in D_{n-1} the H-class H_e containing e is isomorphic (as a group) to the H-class H_f. Thus H_e contains an r-potent π_e such that $\pi_e^{r-1}=e$. Therefore it has been shown that every element of $p^2(D_{n-1})=\{x\in D_{n-1} : xx=x\}$ is a power of an element of $p^r(D_{n-1})=\{x\in D_{n-1} : x \text{ is an } r\text{-potent}\}$. It follows from [2] that every element of D_{n-1} is a product of elements of $p^r(D_{n-1})$.

(3) We have that $D_{i-1}\subset D_i D_i$ by [3, Lemma 4] and hence $p^r(D_{n-1})$ generates the entire semigroup $S_n = D_0 \cup D_1 \cup D_2 \cup \cdots \cup D_{n-1}$ of all elements of T_X of rank less than n. This proves the theorem.

REMARK 2. In [4], the author proved that: Let r be a fixed positive integer with $2<r<n=\dim E$. Let $L(E)$ be the semigroup of all linear mappings of a vector space E into E. Let $S(E)$ be the subsemigroup of $L(E)$ consisting of all singular mappings of E. Then every element of $S(E)$ can be expressed as a finite product of r-potents in $S(E)$

PROBLEM. Count the number of all r-potents in T_X for a fixed positive integer r $(3\leq r\leq n)$ for $X=\{x_1, x_2, \cdots, x_n\}$. (See [3] for this problem.)

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