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ON THE SUM OF THE LARGEST k-TH DIVISORS

By S. M. Lee

There have been long and extensive studies about topics related to the sum and number of divisor of an integer n [2]. The purpose of this research is to study the following problem:

Let $k>1$ be a positive integer. For each positive n, let $\delta_k(n)$ be the largest integer such that $(\delta_k(n))^k$ divides n. For any positive real number x, let

$$D_k(x)=\sum_{n\leq x}\delta_k(n).$$

We are going to investigate the order of $D_k(x)$.

LEMMA. $$\delta_k(n)=\sum_{d^k|n}\varphi(d)$$

PROOF. By the characteristic property of Euler's φ-function, in particular [3, Theorem 262]

$$n=\sum_{d|n}\varphi(d).$$

We obtain

$$\delta_k(n)=\sum_{d|\delta_k(n)}\varphi(d)=\sum_{d^k|n}\varphi(d),$$

since by the fundamental theorem of arithmetic, $d^k|n$ if and only if $d|\delta_k(n)$.

First we give several crude estimates of $D_k(x)$.

THEOREM 1.

$$D_k(x)=o(x^{\frac{1}{k}+1}).$$

PROOF.

$$D_k(x)=\sum_{n\leq x}\delta_k(n)\leq\sum_{n\leq x}n^{\frac{1}{k}}$$
$$\leq\sum_{n\leq x}x^{\frac{1}{k}}=x^{\frac{1}{k}}\sum_{n\leq x}1\leq x^{\frac{1}{k}+1}.$$

THEOREM 2.

$$D_k(x)=\begin{cases}o(x) & \text{for } k\neq 2.\\ o(x\log x) & \text{for } k=2.\end{cases}$$

PROOF.

$$\begin{aligned}D_k(x) &= \sum_{n\le x}\delta_k(n)=\sum_{n\le x}\sum_{d^k|n}\varphi(d)\\ &\le \sum_{n\le x}\sum_{d^k e=n} d=\sum_{d\le x} d\sum_{e\le x/d^k} 1\\ &\le \sum_{d\le x} d\frac{x}{d^k}=x\sum_{d\le x}\frac{1}{d^{k-1}}\\ &=\begin{cases} o(x) & \text{if } k>2.\\ o(x\log(x)) & \text{if } k=2.\end{cases}\end{aligned}$$

COROLLARY. *For $\alpha>1$ we have*

$$\lim_{x\to\infty}\frac{D_2(x)}{x(\log x)^\alpha}=0$$

and

$$\lim_{n\to\infty}\frac{D_k(x)}{x(\log x)^\alpha}=0 \quad \textit{for } \alpha>0 \textit{ and } k>2.$$

We establish a lower estimate.

THEOREM 3. *For large x, we have*

$$D_k(x)\ge\frac{x}{4(k-1)}.$$

PROOF.

$$\begin{aligned}D_k(x) &= \sum_{n\le x}\sum_{d^k e=n}\varphi(d)=\sum_{d\le x^{1/k}}\varphi(d)\left[\frac{x}{d^k}\right]\\ &\ge\frac{1}{2}\sum_{d\le x^{1/k}}\varphi(d)\ \frac{x}{d^k}=\frac{x}{2}\sum_{d\le x^{1/k}}\frac{\varphi(d)}{d^k}\\ &\ge\frac{x}{2}\sum_{d\le x^{1/k}}\frac{1}{d^k}\ge\frac{x}{2}\int_1^{x^{1/k}}\frac{dt}{t^k}\\ &=\frac{x}{2}\left(\frac{1}{-k+1}+\frac{1}{-k+1}\,\frac{1}{x^{(k-1)/k}}\right)\\ &\ge\frac{x}{2}\left(\frac{1}{k-1}-\frac{1}{2(k-1)}\right)=\frac{x}{4(k-1)} \quad \text{for } x \text{ large.}\end{aligned}$$

The following theorem is our main result. We need the the following formula.

LEMMA. [1, p. 104] *For real $s>2$,*

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s}=\frac{\zeta(s-1)}{\zeta(s)}.$$

THEOREM 4. *Let $k \geq 3$ be any integer. Then*

$$D_k(x)=\left(\frac{\zeta(k-1)}{\zeta(k)}\right)x+o(x^{2/k}).$$

PROOF. Let $x=x_1^k x_2$, $x_1 \geq 1$, and $x_2 \geq 1$. Then

$$D_k(x)=\sum_{n\leq x}\delta_k(n)=\sum_{n\leq x}\sum_{d^k|n}\varphi(d)=\sum_{d^k e\leq x}\varphi(d)$$

$$=\sum_{\substack{d^k e\leq x\\ d\leq x_1}}\varphi(d)+\sum_{\substack{d^k e\leq x\\ e\leq x_2}}\varphi(d)-\sum_{\substack{d^k e\leq x\\ d\leq x_1, e\leq x_2}}\varphi(d).$$

We denote the three sums by $\sum_1$, $\sum_2$, $\sum_3$, respectively.

$$\sum\nolimits_1=\sum_{\substack{d^k e\leq x\\ d\leq x_1}}\varphi(d)=\sum_{d\leq x_1}\varphi(d)\left[\frac{x}{d^k}\right]$$

$$=\sum_{d\leq x_1}\varphi(d)\left(\frac{x}{d^k}+o(1)\right)$$

$$=x\sum_{d\leq x_1}\frac{\varphi(d)}{d^k}+o\Big(\sum_{d\leq x_1}\varphi(d)\Big)$$

$$=x\left(\frac{\zeta(k-1)}{\zeta(k)}\right)-xo(x_1^{2-k})+o(x_1^2)$$

$$=x\left(\frac{\zeta(k-1)}{\zeta(k)}\right)+o(x_1^2x_2).$$

$$\sum\nolimits_2=\sum_{\substack{d^k e\leq x\\ e\leq x_2}}\varphi(d)=\sum_{d^k\leq x}\varphi(d)\sum_{e\leq x_2/d^k}1\leq x_2\sum_{d^k\leq x}\frac{\varphi(d)}{d^k}$$

$$\leq x_2\sum_{d=1}^{\infty}\frac{\varphi(d)}{d^k}=o(x_2).$$

$$\sum\nolimits_3=\sum_{\substack{d\leq x_1\\ e\leq x_2}}\varphi(d)=\Phi(x_1)[x_2]$$

$$=\left(\frac{3x_1^2}{\pi^2}+o(x_1^{2-\delta})\right)[x_2]$$

$$=o(x_1^2x_2).$$

Collecting $\sum_1$, $\sum_2$, $\sum_3$, it follows that

$$D_k(x)=\left(\frac{\zeta(k-1)}{\zeta(k)}\right)x+o(x_1^2x_2)$$

Now, let $x_1=x^a$, $x_2=x^b$, such that $x_1^k x_2=x$, and $x_1\geq 1$, $x_2\geq 1$. Then

$$x_1^2x_2=x^{2a+b}$$

Choose $a=1/k$ and $b=0$. Then $ka+b=1$, $a>0$, $b\geq 0$, and $2a+b=2/k<1$. Therefore

$$D_k(x)=\left(\frac{\zeta(k-1)}{\zeta(k)}\right)x+o(x^{2/k}).$$

We also wish to show that $2/k$ is the smallest number for $2a+b$ with $ka+b=1$ for $k\geq 3$, $a>0$, and $b\geq 0$. Let $\varepsilon=2a+b$. Then

$$a=\frac{1-\varepsilon}{k-2} \text{ and } b=\frac{k\varepsilon-2}{k-2}.$$

Since $b\geq 0$ and $k>2$, we have $k\varepsilon-2\geq 0$. That is, $\varepsilon\geq 2/k$. Also when $a=1/k$, $b=0$, $2a+b=2/k$. Therefore $2/k$ is the smallest number for $2a+b$ with $ka+b=1$ for $k\geq 3$, $a>0$, and $b\geq 0$.

COROLLARY.

$$D_3(x)=\left(\frac{\pi^2}{6\zeta(3)}\right)x+o(x^{2/k})$$

COROLLARY.

$$\lim_{x\to\infty}\frac{D_3(x)}{x}=\frac{\pi^2}{6\zeta(3)}$$

and

$$\lim_{x\to\infty}\frac{D_k(x)}{x}=\frac{\zeta(k-1)}{\zeta(k)}$$

for $k\geq 3$.

Kansas State University
Manhattan, Kansas 66502

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