Kyungpook Math. J.
Volume 15, Number 1
June, 1975

ABIAN'S ORDER RELATION ON $C(X)$

By W.D. Burgess*

1. Introduction

It was observed by Abian (see, for example [1]) that the natural order on a Boolean ring can be defined for reduced rings (a ring in which zero is the only nilpotent) by writing "$a \leq b$ iff $ab=a^2$". The relation makes a reduced ring a partially ordered multiplicative semigroup. This relation was studied by Abian [1] and Chacron [6] to characterize products of division rings. Burgess and Raphael ([4] and [5]) studied order completions of these rings. This article will extend considerably the class of rings where Abian's order is well understood. For the convenience of the reader certain definitions and results from the above-mentioned articles will be quoted. To simplify, all rings referred to will be assumed *commutative* and *semiprime with* 1.

Two kinds of completeness were studied in [4] and [5].

DEFINITION 1. A subset X of a ring R is called *orthogonal* if for all $a, b \in X$, $a \neq b$, $ab=0$. R is called *orthogonally complete* if every orthogonal set in R has a supremum. An extension $R \subset S$ is an *orthogonal completion* if S is orthogonally complete and every element of S is the supremum of an orthogonal set in R.

DEFINITION 2. A subset X of R is called *boundable* if for all $a, b \in X$, $ab(a-b)=0$. R is called *complete* if every boundable set in R has a supremum. An extension $R \subset S$ is a *completion* if S is complete and every element of S is the supremum of a boundable set in R.

The word "boundable" was chosen since boundable sets in R are exactly those sets which have upper bounds in some extension. Also, in a Boolean ring every set is boundable. Now it is shown in [5] that the two kinds of completeness and completions coincide if R has enough idempotents; more exactly when R is i-dense. A ring R is *i-dense* if every idempotent of its complete ring of quotients, $Q(R)$, is the supremum of a set of idempotents in R. The class of i-dense rings includes

*The author acknowledges the support of grant A 7539 of the National Research Council of Canada. Also, he is grateful to his topologist colleagues, especially L. Demers and M. O'Keefe, for their assistance.

the classes of Baer and of regular rings (even all p.p. rings and more). All i-dense rings have (orthogonal) completions and these are rings of quotients with respect to a topologizing family of large ideals. In general rings do not have completions or orthogonal completions.

Other useful facts are the following. An (orthogonal) completion of a ring R is unique, if it exists, and is a subring of the complete ring of quotients, $Q(R)$, of R ([4]). Also $Q(R)$ is always complete [4]. For the ring of continuous functions $C(X)$, we write $Q(X)$ for $Q(C(X))$. $Q(X)$ may be expressed [7] as the ring of equivalence classes of continuous functions defined on dense open subsets of X. The terminology of rings of continuous functions is that of [7] and [8]. In what follows, $C(X)$ always refers to the ring of continuous real-valued functions on a *completely regular* space X.

2. Completions

It should be emphasized that, throughout, the order relation referred to is not the natural order in $C(X)$. We do have $f\leq g$ implies $|f|\leq|g|$, but not conversely.

A subset $\{f_\alpha\}$ of $C(X)$ is orthogonal if for $\alpha\neq\beta$, $\operatorname{coz} f_\alpha \cap \operatorname{coz} f_\beta=\phi$ ($\cos f_\alpha=\{x\mid f_\alpha(x)\neq 0\}$, $z(f_\alpha)$ is its complement). Also $\{f_\alpha\}$ is boundable if for f_α, f_β we have: for all $x\in X$, $f_\alpha(x)\neq 0$ and $f_\beta(x)\neq 0$ imply $f_\alpha(x)=f_\beta(x)$.

The first thing to observe here is that since every boundable set in $C(X)$ has a supremum in $Q(X)$, this is an equivalence class of continuous functions defined on dense open subsets of X. An explicit representative may be found.

PROPOSITION 3. *If $\{f_\alpha\}_{\alpha\in A}$ is a boundable set in $C(X)$, then the function q defined by*

$$q(x)=\begin{cases} f_\alpha(x) & \text{if } x\in\operatorname{coz} f_\alpha \\ 0 & \text{if } x\in\operatorname{Int}(\bigcap_\alpha z(f)) \end{cases}$$

represents the supremum, ϕ, of $\{f_\alpha\}$ in $Q(X)$.

PROOF. Since $\{f_\alpha\}$ is boundable, if $x\in\operatorname{coz} f_\alpha\cap\operatorname{coz} f_\beta$ then $f_\alpha(x)=f_\beta(x)$; hence q is well-defined. It is continuous on the dense open set $\bigcup_\alpha\cos f_\alpha \cup \operatorname{Int}(\bigcap_\alpha z(f_\alpha))=V$, let it represent $\phi\in Q(X)$.

For each $x\in V$, $f_\alpha(x)q(x)=f_\alpha(x)^2$ so ϕ is an upper bound of $\{f_\alpha\}$. Let $\psi\in Q(X)$ be an upper bound of $[f_\alpha]$ in $Q(X)$ represented by h defined on the dense open set W. For each α, $f_\alpha h-f_\alpha^2$ is zero on some dense open set V_α. For $x\in V\cap W$, $(g\ h)(x)=q(x)f_\beta(x)=q(x)^2$ if $x\in$ coz $f_\alpha\cap V_\beta$ and $q(x)h(x)=0=q(x)^2$ if $x\in\operatorname{Int}(\bigcap_\alpha$

$z(f_\alpha))\cap V_\beta$. Hence $qh-q^2$ is zero on the dense open set

$$\bigcup_\beta[\cos f_\beta \cup \mathrm{Int}(\bigcap_\alpha z(f_\alpha))\cap V_\beta].$$

As will be seen later $C(X)$ does not often have a completion, but it does if it is i-dense.

PROPOSITION 4. *$C(X)$ is i-dense if, and only if, every non-empty open set of X contains a non-empty clopen (closed and open) set.*

PROOF. By [5, 4 Proposition] $C(X)$ is i-dense iff every non-zero annihilator ideal contains a non-zero idempotent.

Suppose $C(X)$ is i-dense. Let $\phi\neq U$ be open in X. Since X is completely regular the zero set neighbourhoods form a basis for the neighbourhoods of each point [8 p. 38]. For $x\in U$ let $z(f)$ be a zero set neighbourhood of x in U. Now $g\in\mathrm{Ann}\{f\}$ means $\mathrm{coz}\, g\subset z(f)$. By complete regularity $\mathrm{Ann}\{f\}\neq 0$ so for some idempotent $e\neq 0$, $\mathrm{coz}\, e\subset z(f)$.

Conversely, if $I=\mathrm{Ann}\, S$ is a non-zero annihilator, $\bigcup_{f\in S}\mathrm{coz}\, f$ is not dense. Hence there is a non-empty clopen set E in $\sim\mathrm{Cl}\,(\bigcup_{f\in S}\mathrm{coz}\, f)$. Define e by

$$e(x)=\begin{cases}0 & x\notin E\\ 1 & x\in E.\end{cases}$$

All 0-dimensional spaces (such as p-spaces and extremally disconnected spaces) satisfy the conditions of the proposition. However it is not necessary that X be 0-dimensional in order that $C(X)$ be i-dense. Consider, for example,

$$X=\{(x,y)\in R^2\,|\,0\leq x\leq 1,\ 0\leq y\leq 1,\ x,y \text{ rational if } y\neq 0\}$$

with the subspace topology. $C(X)$ is i-dense but X is not 0-dimensional.

PROPOSITION 5. *If $C(X)$ is i-dense, $Q(X)$ is the (orthogonal) completion of $C(X)$.*

PROOF. It suffices to show that each element of $Q(X)$ is the supremum of an orthogonal set in $C(X)$. Let $\phi\in Q(X)$ be represented by q continuous on a dense open set V. Let $\mathscr{B}$ be a maximal family of disjoint clopen sets in V. By Proposition 4, $\bigcup_{U\in\mathscr{B}}U$ is dense in V. Define, for each $U\in\mathscr{B}$, q_U by $q_U(x)=\begin{cases}q(x) & \text{if } x\in U\\ 0 & \text{if } x\notin U.\end{cases}$ Since U is clopen and q is continuous, $q_U\in C(X)$. The set $\{q_U\}_{U\in\mathscr{B}}$ is orthogonal and by Proposition 3, ϕ is its supremum.

We next show that $C(X)$ is never orthogonally complete if X has a non-trivial connected component. The proof which follows is due to M. O'Keefe and another, using βX, was supplied to us by L. Demers.

PROPOSITION 6. *If X has a non-trivial connected component, then $C(X)$ is not orthogonally complete.*

PROOF. Suppose $\{k_\alpha\}_{\alpha\in A}$ is an orthogonal set in $C(X)$. If $C(X)$ were complete there would be a function $q\in C(X)$, $q=\sup\{k_\alpha\}$. Hence for $x\in \operatorname{coz} k_\alpha$, $q(x)=k_\alpha(x)$ and for $x\in \operatorname{Int}(\bigcap_\alpha z(k_\alpha))$, $q(x)=0$; since q would coincide with the function defined in Proposition 3.

Let $u\neq v$ be points of a connected component Y of X and U a closed neighbourhood of u not containing v. There exists $f\in C(X)$ so that $f(U)=0$ and $f(v)=1$. Now f takes on all values in $[0,1]$ and we may assume that for all $x\in X$, $0\leq f(x)\leq 1$. Consider the closed sets $F_n=f^{-1}\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right)$, $n=1, 2, \cdots$. We define two orthogonal families of continuous functions as follows.

For $n\geq 2$, define

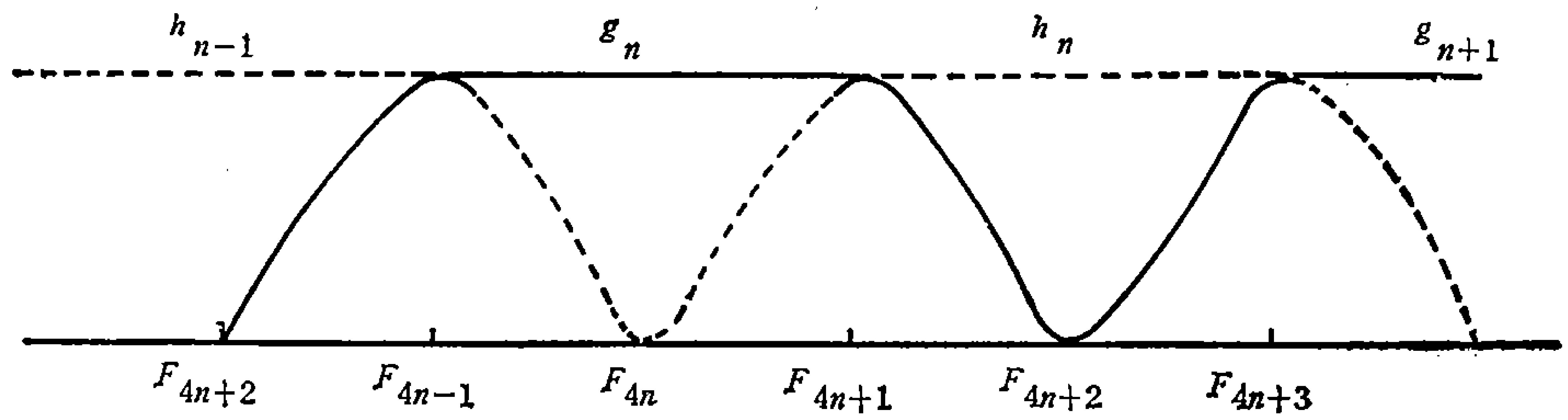
$$g_n(x)=\begin{cases} \frac{1}{f(x)}-(4n-2) & x\in F_{4n-2} \\ 1 & x\in F_{4n-1}\cup F_{4n} \\ 4n+2-\frac{1}{f(x)} & x\in F_{4n+1} \\ 0 & \text{otherwise;} \end{cases}$$

$$h_1(x)=\begin{cases} 1 & x\in F_1\cup\cdots\cup F_6 \\ 8-\frac{1}{f(x)} & x\in F_7 \\ 0 & \text{otherwise,} \end{cases}$$

and, for $n\geq 2$,

$$h_n(x)=\begin{cases} \frac{1}{f(x)}-4n & x\in F_{4n} \\ 1 & x\in F_{4n+1}\cup F_{4n+2} \\ 4n+4-\frac{1}{f(x)} & x\in F_{4n+3} \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively:



If $C(X)$ were orthogonally complete, both $\{g_n\}$ and $\{h_n\}$ would have suprema in $C(X)$, say g and h, respectively. Also $g+h$ would be a continuous function. But for $x \in F_n$, $g(x)+h(x) \geq 1$ and for $x \in \mathrm{Int}\ z(f)$, $g(x)=h(x)=0$. We shall show that this contradicts the continuity of $g+h$. To see this note that Int $z(f)$ is not clopen since it meets the component Y. Hence there is $x \in \mathrm{Cl\ Int}\ z(f)$, $x \notin \mathrm{Int}\ z(f)$. Then, any neighbourhood of x meets both Int $z(f)$ and the exterior of $z(f)$. But $g+h$ is zero on Int $z(f)$ and greater than or equal to 1 on its exterior.

We next show that $C(X)$ does not have a completion (or orthogonal completion) if X is locally connected and has a non-trivial component.

LEMMA 7. *A product of rings $\prod R_\alpha$ has a completion (orthogonal completion) if, and only if, each R_α does. In this case the (orthogonal) completion is the product of the (orthogonal) completions of the R_α.*

PROOF. It suffices to observe that suprema in $Q(\prod R_\alpha)=\prod Q(R_\alpha)$ are computed "componentwise".

LEMMA 8. *If X is a non-trivial connected and locally connected space then $C(X)$ does not have either a completion or an orthogonal completion.*

PROOF. Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a boundable family from $C(X)$ with more than one non-zero element. Then if f_α, $f_\beta \neq 0$, $\alpha \neq \beta$, $f_\alpha f_\beta (f_\alpha - f_\beta)=0$ so coz $f_\alpha \neq X$. By connectedness the open set coz f_α is not closed and so coz f_α has a boundary point x. For each $\varepsilon > 0$, a connected neighbourhood of x will contain points y so that $0 < |f_\alpha(y)| < \varepsilon$. Hence a representative of sup $\{f_\alpha\}$ is never bounded away from 0.

By Proposition 6, we know that $C(X)$ is neither complete nor orthogonally complete. Hence there is an orthogonal family $\{f_\alpha\}_{\alpha \in \Lambda}$ with no supremum in $C(X)$. Let q be the representative of the supremum, ϕ, as in Proposition 3. Then $r=|q|$ represents sup $\{|f_\alpha|\}$. If $\bar{r} \in Q(X)$ were in $C(X)$ then there would be a continuous extension s of r to X. Define q' by

$$q' = \begin{cases} q(x) & x \in \text{domain of } q \\ 0 & \text{otherwise.} \end{cases}$$

Clearly q' is continuous on D. For $x \notin D$, $x \in \mathrm{Cl} \bigcup_\alpha \mathrm{coz}\ f_\alpha$ and $x \notin \bigcup_\alpha \mathrm{coz}\ f_\alpha$. For $\varepsilon > 0$, there is a connected neighbourhood N of x such that for all $y \in N$, $|s(y)| < \varepsilon$. But for $y \in D \cap N$, $s(y)=|q(y)|$. Hence q' is continuous, a contradiction.

Hence we may assume that $\{f_\alpha\}$ is a family of non-negative functions. If $C(X)$ had a completion (orthogonal completion), S, S would contain $\phi+1$ (1 the constant function) and $\phi+1\notin C(X)$. But any representative of it, such as $q+1$, is bounded away from zero and so could not be the supremum of a set from $C(X)$.

PROPOSITION 9. *Let X be locally connected with a non-trivial component. Then $C(X)$ has no (orthogonal) completion.*

PROOF. Write $X=\bigcup X_\alpha$ as the union of its components. The components are, here, clopen sets. Then $C(X)=\prod C(X_\alpha)$ and by Lemmas 7 and 8, $C(X)$ has no (orthogonal) completion.

3. Conditional completeness.

A partially ordered set is called *conditionally complete* if every bounded subset has a supremum. In this section we mention without proof the following.

PROPOSITION 10. *If X is locally connected then $C(X)$ is conditionally complete.*

The proof uses techniques similar to those used in Section 2.

University of Ottawa
Ottawa, Canada.

REFERENCES

[1] Abian, A., *Direct product decompositions of commutative semisimple rings*, Proc. A.M.S., 24 (1970), pp. 502—507.

[2] Abian, A., *Order relation in rings without nilpotent elements*, typescript, 1973.

[3] Bourbaki, N., *Topologie Générale*. Chapitres 1 et 2. Paris: Hermann, 1961.

[4] Burgess, W.D. and Raphael, R., *Abian's order relation and orthogonal completions for reduced rings*, to appear, Pacific J. Math., 48.

[5] Burgess, W.D. and Raphael, R., *Complete and orthogonally complete rings*, to appear, Canad. Jour. Math.

[6] Chacron, M., *Direct product of division rings and a paper of Abian*, Proc. A.M.S., 29 (1971), pp. 259—262.

[7] Fine, N., Gillman, L. and Lambek, L., *Rings of quotients of rings of functions*. Montreal: McGill University Press, 1965.

[8] Gillman, L. and Jerison, M., *Rings of continuous functions*. Princeton, New Jersey: van Nostrand, 1960.

[9] Lambek, J., *Lectures on rings and modules*. Waltham, Mass: Ginn, 1966.

[10] Rakhra, M., *Abian's order relation and $C(X)$*, thesis, University of Ottawa, 1973.

[11] Willard, S., *General topology*. Reading, Mass: Addison Wesley, 1970.