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INTEGRABILITY THEOREMS FOR TRIGONOMETRIC SERIES

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1. Let $f(x)$ be a function defined by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx, \tag{1.1}$$

where $\{a_n\}$ is a sequence of positive coefficients.

We define a non-negative function $\Phi(x)=x\phi(x)$ such that $\Phi(x)/x^{1+\delta}$ is increasing when x is increasing from zero to infinity, where $\delta>0$ is a sufficiently small constant depending on Φ and that there exists a positive constant $k>1$, depending on Φ, such that $\Phi(x)/x^k$ is decreasing when x is increasing from zero to infinity. It follows that $\Phi'(x)$ exists almost everywhere in any finite interval and that

$$(1+\delta)\ \phi(x)\ <\Phi'(x)\ <k\ \phi(x). \tag{1.2}$$

Suppose further that $\Psi(x)=x\psi(x)$ is a non-decreasing positive function such that $\Psi(x)/x^{1-\delta'}$ is decreasing as x is increasing in $(0,\pi)$ for a sufficiently small constant $\delta'>0$ depending on Ψ. It is clear that $\Psi'(x)>0$, where $\Psi'(x)$ exists almost everywhere in $(0,\pi)$ and that

$$\Psi'(x)\leq(1-\delta')\ (x). \tag{1.3}$$

Shah [7] and Szász [8] generalised the notion of decreasing null sequence in the form of quasi-monotonic sequence in the following fashion.

A sequence $\{a_n\}$ of positive numbers is said to be be quasi-monotone if

$$a_{n+1}\leq a_n(1+\alpha/n)$$

for some constant $\alpha\geq 0$ and all $n>n_0(\alpha)$. We may suppose α to be an integer. An equivalent definition is that $\{a_n\}$ is quasi-monotone if and only if $n^{-\beta}a_n\downarrow 0$ for some $\beta>0$.

An easy consequence of this definition is that every monotonic decreasing null sequence is also quasi-monotone. However, the converse need not be true.

2. Concerning the integrability of trigonometric series for L^p class, Hardy and Littlewood [5, 6] established the following classical theorems.

THEOREM A. *If* $a_n \downarrow 0$ *and* $1<p<\infty$, *then*

$$f(x) \in L^p(0, \pi)$$

if and only if

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

The theorem holds for sine series also.

THEOREM B. *If* $f(x) \geq 0$ *and* f *decreases,* $1<p<\infty$ *and* a_n *are the Fourier cosine coefficients of* f, *then*

$$\sum_{n=0}^{\infty} |a_n|^p < \infty$$

if and only if

$$x^{p-2} f(x)^p \in L(0, \pi).$$

A similar result holds for sine series.

In 1956, extending Theorem A, Chen [2] proved the following theorem.

THEOREM C. *Suppose that* $a_n \downarrow 0$ *and that* $f(x)$ *is defined by* (1.1). *Then for* $p>1, 0<\gamma<1$,

$$x^{-\gamma} [f(x)]^p \in L(0, \pi)$$

if and only if

$$\sum_{n=1}^{\infty} n^{\gamma+p-2} a_n^p < \infty.$$

The result holds for sine series also. He further observed that Theorem C remains true even if a_n is ultimately positive and decreases steadily to zero as n tends to infinity.

Later on, Chen [4] proved the following theorem which generalizes not only the power-function multipliers but also the L^p classes.

THEOREM D. *Let* $f(x)$ *be defined by* (1.1), *where* $a_n \downarrow 0$. *Then a necessary condition for*

$$\Phi[|f(x)|]/\Psi(x) \in L(0, \pi)$$

is that

$$\sum_{n=1}^{\infty} \frac{\Phi(na_n)}{n^2 \Psi(1/n)} < \infty.$$

It may be mentioned here that the sufficiency part of this theorem also holds [3], since the conditions imposed on Φ imply that $\Phi(x)/x$ is increasing as x increases from zero to infinity.

3. In the present paper we relax the condition $a_n \downarrow 0$ of Theorem D by assuming only the quasi-monotonicity of $\{a_n\}$. We also prove the sufficiency of Theorem D under a weaker hypothesis. In what follows we shall prove the following theorems.

THEOREM 1. *Let $\{a_n\}$ be a positive sequence such that $\{n^{-\beta}a_n\}$ is monotonically decreasing for some non-negative integer β. Then a necessary condition for*

$$\Phi[|f(x)|]/\Psi(x)\in L(0,\pi)$$

is that

$$\sum_{n=1}^{\infty}\frac{\Phi(na_n)}{n^2\Psi(1/n)}<\infty,$$

where

$$f(x)\sim\sum_{n=1}^{\infty}a_n\cos nx.$$

THEOREM 2. *Let $\{a_n\}$ be a positive, null sequence such that $\{n^{-\beta}a_n\}$ is monotonically decreasing for some non-negative integer β. If*

$$\sum_{k=n}^{\infty}|a_k-a_{k+1}|\leq K^* a_n, \tag{3.1}$$

$$\sum_{n=1}^{\infty}\frac{\Phi(na_n)}{n^2\Psi(1/n)}<\infty, \tag{3.2}$$

then

$$\Phi[|f(x)|]/\Psi(x)\in L(0,\pi),$$

where

$$f(x)\sim\sum_{n=1}^{\infty}a_n\cos nx.$$

It is clear that if $a_n\downarrow 0$, then $\{a_n\}$ is a quasi-monotonic sequence and that (3.1) holds. Hence Theorem 2 is a generalization of the sufficiency part of Theorem D.

4. We shall require the following lemmas for the proofs of theorems.

LEMMA 1 [3]. *Let $\Phi(x)$ be the function defined in* § 1 *and let $a_k\geq 0$. Then*

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*K denotes a positive constant not necessarily the same at each occurence.

$$\sum_{n=1}^{\infty} \frac{\Phi(\sum_{k=1}^{n} a_k)}{\Phi(n)} \leq K(\Phi) \sum_{n=1}^{\infty} \frac{\Phi(na_n)}{\Phi(n)},$$

where $K(\Phi)$ is a constant depending on Φ.

LEMMA 2 [4]. *Let $a \geq 0$,*

$$F(x) = \int_0^x f(t)\, dt \quad (f(t) \geq 0).$$

Then

$$\int_0^a \frac{\Phi[F(x)/x]}{\Psi(x)}\, dx \leq K(\Phi) \int_0^a \frac{\Phi[f(x)]}{\Psi(x)}\, dx,$$

where $K(\Phi)$ is a positive constant, depending on Φ.

LEMMA 3. *Let $a \geq 0$,*

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt \qquad (f(t) \geq 0).$$

Then

$$\int_0^a \frac{\Phi[F(x)/x]}{\Psi(x)}\, dx \leq K(\Phi) \int_0^a \frac{\Phi[f(x)/x]}{\Psi(x)}\, dx.$$

PROOF. Let f_n and F_n be defined by

(4.1) $$f_n(x) = \begin{cases} f(x) & (1/n \leq x \leq a) \\ 0 & (0 \leq x < 1/n) \end{cases}$$

(4.2) $$F_n(x) = \frac{1}{x} \int_0^x f_n(t)\, dt.$$

Integration by parts yields

$$\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)} dx = \frac{a\Phi[F_n(a)/a]}{\Psi(c)} + \int_0^a \frac{F_n(x)\Phi'[F_n(x)/x]}{x\Psi(x)}\, dx$$

$$+ \int_0^a \frac{x\Psi'(x)\Phi[F_n(x)/x]}{[\Psi(x)]^2}\, dx$$

$$- \int_0^a \frac{F_n(x)\Phi'[F_n(x)/x]}{\Psi(x)}\, dx.$$

Differentiating (4.2) with respect to x we have

$$F_n'(x)=\frac{f_n(x)}{x}-\frac{F_n(x)}{x}. \tag{4.3}$$

Therefore by (4.1), (4.2) and (4.3) we have

$$\begin{aligned}\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)}dx=2\int_0^a \frac{F_n(x)\Phi'[F_n(x)/x]}{x\Psi(x)}dx \\ +\int_0^a \frac{x\Psi'(x)\Phi[F_n(x)/x]}{[\Psi(x)]^2}dx \\ -\int_{1/n}^a \frac{f(x)\Phi'[F_n(x)/x]}{x\Psi(x)}dx.\end{aligned}$$

Hence by the definition of $\Phi(x)$ and (1.2) it follows that

$$\begin{aligned}\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)}dx\geq 2(1+\delta)\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)}dx \\ -k\int_{1/n}^a \frac{f(x)\phi[F_n(x)/x]}{x\Psi(x)}dx.\end{aligned} \tag{4.4}$$

Let $t\geq 1$. Then, $\phi(x)$ being increasing, we have

$$\begin{aligned}&\frac{f(x)}{x}\phi\{F_n(x)/x\} \\ &=t^{-1}\left[t\frac{f(x)}{x}\phi\{F_n(x)/x\}\right] \\ &\leq t^{-1}\operatorname{Max}\left[t\frac{f(x)}{x}\phi\{tf(x)/x\},\ \phi\{F_n(x)/x\}\{F_n(x)/x\}\right] \\ &\leq t^{-1}\left[\Phi(tf(x)/x\}+\Phi\{F_n(x)/x\}\right] \\ &\leq t^{k-1}\Phi[f(x)/x]+t^{-1}\Phi[F_n(x)/x].\end{aligned} \tag{4.5}$$

Thus, by virtue of (4.5) we obtain

$$(1+2\delta-kt^{-1})\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)}dx\leq kt^{k-1}\int_0^a \frac{\Phi[f(x)/x]}{\Psi(x)}dx.$$

Now taking an arbitrary fixed and sufficiently large value of t, in order to make $1+2\delta-kt^{-1}>0$, we observe that

$$\int_0^a \frac{\Phi[F_n(x)/x]}{\Psi(x)}dx\leq K(\Phi)\int_0^a \frac{\Phi[f(x)/x]}{\Psi(x)}dx.$$

Since $K(\Phi)$ is independent of n, taking the superior limit on the left we have

$$\int_0^a \frac{\Phi[F(x)/x]}{\Psi(x)}\,dx \leq K(\Phi)\int_0^a \frac{\Phi[f(x)/x]}{\Psi(x)}\,dx.$$

Thus the lemma 3 is proved.

LEMMA 4. *Let* $\Phi[|f(x)|]/\Psi(x)\in L(0,\pi)$, *where*

$$f(x)\sim\sum_{n=1}^{\infty} a_n \cos nx.$$

Assume that the Fourier coefficients a_n *of* f *are non-negative. Define*

$$A(n)=\sum_{j=[n/2]}^{n} a_j. \tag{4.6}$$

Then

$$\sum_{n=1}^{\infty}\frac{\Psi[A(n)]}{n^2\Psi(1/n)}<\infty. \tag{4.7}$$

PROOF. Without any loss of generality we assume that $a_0=0$. Let

$$f_1(x)=\int_0^x f(u)\,du,$$

$$f_2(x)=\int_0^x f_1(u)\,du.$$

Then by virtue of integration of Fourier series of f,

$$f_2(x)=\sum_{j=1}^{\infty} a_j\,(1-\cos jx)\,j^{-2}$$

$$\geq\sum_{j=[n/2]}^{n} a_j(1-\cos jx)j^{-2},$$

for any integer n.

Using the inequality $(1-\cos nx)\geq\frac{K}{2}(nx)^2$ for $\pi/[4(n+1)]\leq x\leq\pi/(4n)$, we obtain

$$A(n)\leq Kn^2 f_2(x). \tag{4.8}$$

Then, by means of lemmas 2 and 3, it follows that

$$\sum_{n=1}^{\infty}\frac{\Phi[A(n)]}{n^2\Psi(1/n)}\leq K\sum_{n=1}^{\infty}\frac{\Phi[n^2 f_2(x)]}{n^2\Psi(1/n)}$$

$$\leq K\sum_{n=1}^{\infty}\int_{\pi/[4(n+1)]}^{\pi/(4n)}\frac{\Phi[x^{-2}f_2(x)]}{\Psi(x)}\,dx$$

$$=K\int_0^{\pi/4}\frac{\Phi[x^{-2}\int_0^x f_1(u)du]}{\Psi(x)}dx$$

$$\leq K(\Phi)\int_0^{\pi/4}\frac{\Phi[x^{-1}|f_1(x)|]}{\Psi(x)}dx$$

$$=K(\Phi)\int_0^{\pi/4}\frac{\Phi[x^{-1}\int_0^x|f(u)|du]}{\Psi(x)}dx$$

$$\leq K(\Phi)\int_0^{\pi/4}\frac{\Phi[|f(x)|]}{\Psi(x)}dx$$

$$<\infty.$$

This completes the proof of Lemma 4.

5. Proof of Theorem 1

Since $\{n^{-\beta}\ a_n\}$ is monotonically decreasing, we have

$$a_n=a_n n^{-\beta}n^{\beta}$$

$$\leq Kn^{\beta-1}\sum_{j=[n/2]}^{n} j^{-\beta}\ a_j$$

$$\leq Kn^{-1}\sum_{j=[n/2]}^{n} a_j$$

$$=Kn^{-1}\ A(n).$$

Hence, by virtue of Lemma 4, we have

$$\sum_{n=1}^{\infty}\frac{\Phi(na_n)}{n^2\Psi(1/n)}\leq K\sum_{n=1}^{\infty}\frac{\Phi[A(n)]}{n^2\Psi(1/n)}$$

$$<\infty.$$

Theorem 1 is thus established.

6. Proof of Theorem 2

The condition (3.1) implies that $\{a_n\}$ is sequence of bounded variation. Hence the Fourier series of f converges for $x>0$ [9, Vol. 1, p. 4]. That is,

$$f(x)=\sum_{k=1}^{\infty}a_k\cos kx$$

$$=\sum_{k=1}^{\infty}a_k\cos kx+\sum_{k=n+1}^{\infty}a_k\cos kx.$$

If $D_n(x)$ denotes the Dirichlet's kernel, then using partial summation we obtain

$$\sum_{k=n+1}^{\infty} a_k \cos kx = \sum_{k=n}^{\infty} (a_k - a_{k+1}) D_k(x) - a_n D_n(x).$$

Hence, for any integer n, we have

$$|f(x)| \leq S_n + \sum_{k=n}^{\infty} |a_k - a_{k+1}| \ |D_k(x)| + |a_n| \ |D_n(x)|$$

$$= S_n + o(1/x) \sum_{k=n}^{\infty} |a_k - a_{k+1}| + o(x^{-1} a_n),$$

where

$$S_n = \sum_{k=1}^{n} a_k.$$

Then, on account of (3. 1), it follows that

$$\int_0^{\pi/2} \frac{\Phi[|f(x)|]}{\Psi(x)} dx$$

$$= \sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{\Phi[|f(x)|]}{\Psi(x)} dx$$

$$\leq \sum_{n=2}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \frac{[S_n + o(1/x) o(a_n) + o(1/x) o(a_n)]}{\Psi(x)} dx$$

$$\leq K \sum_{n=2}^{\infty} \frac{\Phi[S_n + na_n]}{n^2 \Psi(1/n)}.$$

Since $\{n^{-\beta} a_n\}$ is a monotonically decreasing sequence, we have

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k-1}^{n} k^{-\beta} a_k k^{\beta}$$

$$\geq n^{-\beta} a_n \sum_{k=1}^{n} k^{\beta}$$

$$\geq K n^{-\beta} a_n n^{\beta+1}$$

$$= K n a_n.$$

Therefore we have

$$\int_0^{\pi/2} \frac{\Phi[|f(x)|]}{\Psi(x)} dx \leq K \sum_{n=2}^{\infty} \frac{\Phi(S_n)}{n^2 \Psi(1/n)}.$$

Now $n^2 \Psi(1/n)$ has the same properties as that of $\Phi(n)$.
Therefore, using 1, we have

$$\int_0^{\pi/2} \frac{\Phi[|f(x)|]}{\Psi(x)} dx \leq K(\Phi) \sum_{n=2}^{\infty} \frac{\Phi(na_n)}{n^2\Psi(1/n)} < \infty.$$

Similarly it can be shown that

$$\int_{\pi/2}^{\pi} \frac{\Phi[|f(x)|]}{\Psi(x)} dx < \infty.$$

This completes the proof of Theorem 2.

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