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THE RADIUS OF CONVEXITY OF SOME REGULAR FUNCTIONS

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1. Introduction and statement of results

Let S denote the classs of functions $f(z)$ regular and univalent in the open unit disk $E=\{z: |z|<1\}$ which are normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let $S(a)$ denote the subclass of functions $f(z)$ in S satisfying

(1.1) $$\mathrm{Re}\{zf'(z)/f(z)\}>a,$$

for all z in E, where $0\leq a<1$. A function $f(z)$ in $S(a)$ is said to be starlike of order a and a function in $S(0)$ is called starlike. Let $K(a)$ denote the subclass of functions $f(z)$ in S satisfying

(1.2) $$\mathrm{Re}\{zf''(z)/f(z)\}+1>a,$$

for all z in E, where $0\leq a<1$. A function $f(z)$ in $K(a)$ is said to be convex of order a and a function in $K(0)$ is called convex.
Let $M(a)$ denote the subclass of functions $f(z)$ in S satisfying

(1.3) $$\left|\frac{zf'(z)}{f(z)}-\frac{1}{2a}\right|<\frac{1}{2a},$$

for all z in E, where $0\leq a<1$. It is clearly that a function in $M(a)$ is starlike and $M(0)$ is the same as the class $S(0)$. The class $M\left(\frac{1}{2}\right)$ has been investigated by R. Singh [2, 3].

In this paper we will prove the following theorems.

THEOREM 1. *If* $0\leq a\leq b<1$, $f(z)$ *is in* $S(a)$, *and*

(1.4) $$f(z)=z+a_{m+1}z^{m+1}+a_{m+2}z^{m+2}+\cdots,$$

$m\geq 1$, *then* $f(z)$ *is convex of order* b *in the region* $|z|<r_0^{1/m}$, *where* r_0 *is the smallest positive root of the equation*

(1.5) $$Q(t)\equiv(4a^2+b+1-4a-2ab)t^2+(4a+2am-2ab-2m-2)t+(1-b)=0.$$

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This result is sharp.

THEOREM 2. *If* $0\leq a,\ b<1$, $f(z)$ *is in* $M(a)$, *and*

(1.6) $$f(z)=z+a_{m+1}z^{m+1}+a_{m+2}z^{m+2}+\cdots,$$

$m\geq 1$, *then* $f(z)$ *is convex of order* b *in the region* $|z|<R^{1/m}$, *where* R *is the smallest positive root of the equation*

(1.7) $$H(t)\equiv(1+bc)t^2-(2+m+mc+bc-b)t+(1-b)=0,\quad c=1-2a.$$

This result is sharp.

2. We need the following lemmas

LEMMA 1. [1, Lemma 1] *If* $p(z)=1+c_m z^m+c_{m+1}z^{m+1}+\cdots$ *is analytic and satisfies* $\mathrm{Re}(p(z))>a$, $0\leq a<1$, $m\geq 1$. *for* $|z|<1$. *Then we have*

(2.1) $p(z)=(1+(2a-1)z^m u(z))/(1+z^m u(z))$, *for* $|z|<1$,

where $u(z)$ *is analytic and* $|u(z)|\leq 1$ *for* $|z|<1$.

LEMMA 2. [1, p. 240 Lemma 2, 3 and inequality (6)].
Under the hypothesis of Lemma 1 *we have for* $|z|<1$,

(2.2) $|zp'(z)/p(z)|\leq 2m|z|^m(1-a)/\ \{(1-|z|^m)[1+(1-2a)|z|^m]\}$.

(2.3) $\mathrm{Re}(p(z))\geq[1+(2a-1)|z|^m]/(1+|z|^m)$, *and*

(2.4) $|zp'(z)|\leq 2m|z|^m[\mathrm{Re}(p(z))-a]/(1-|z|^{2m})$.

LEMMA 3. *Let* $p(z)=1+d_m z^m+d_{m+1}z^{m+1}+\cdots$ *be analytic and satisfy* $\left|p(z)-\frac{1}{2a}\right|<\frac{1}{2a}$, $0\leq a<1$, $m\geq 1$, *for* $|z|<1$. *Then we have for* $|z|<1$

(2.5) $(1-|z|^m)/(1+c|z|^m)\leq\mathrm{Re}(p(z))\leq|p(z)|\leq(1+|z|^m)/(1-c|z|^m)$,

where $c=1-2a$.

PROOF. Since $\left|p(z)-\frac{1}{2a}\right|<\frac{1}{2a}$ if and only if $\mathrm{Re}(1/p(z))>a$.

Hence by Lemma 1, we can write $p(z)$ as

(2.6) $p(z)=(1+z^m u(z))/(1-cz^m u(z))$, where $c=1-2a$, also $-1<c\leq 1$.

we have

$$|(p(z)-1)/(cp(z)+1)|=|z^m u(z)|\leq|z|^m \text{ for } |z|\leq 1.$$

Therefore

$$|p(z)|^2-2\mathrm{Re}(p(z))+1\leq|z|^{2m}(c^2|p(z)|^2+2c\ \mathrm{Re}(p(z))+1),$$

$$\text{or } |p(z)|^2-2\mathrm{Re}(p(z))\frac{1+c|z|^{2m}}{1-c^2|z|^{2m}}+\frac{1-|z|^{2m}}{1-c^2|z|^{2m}}\le 0.$$

After completing the square and simplifying, we have

$$\left|p(z)-\frac{1+c|z|^{2m}}{1-c^2|z|^{2m}}\right|\le\frac{(1+c)|z|^m}{1-c^2|z|^{2m}}.$$

Hence we obtain

$$(1-z^m)/(1+cz^m)\le\mathrm{Re}(p(z))\le|p(z)|\le(1+|z|^m)/(1-c|z|^m).$$

LEMMA 4. *Under the hypothesis of Lemma* 3 *we have for* $|z|<1$

$$|zp'(z)/p(z)|\le(1+c)m|z|^m/[(1+c|z|^m)(1-|z|^m)]. \tag{2.7}$$

PROOF. Logarithmic differentiation both sides of equation $h(z)p(z)=1$, yields

$$\frac{zh'(z)}{h(z)}=-\frac{zp'(z)}{p(z)}.$$

Since $\mathrm{Re}(1/p(z))>a$, therefore $\mathrm{Re}(h(z))>a$, hence from Lemma 2 we get our inequality (2.7).

3. Proof of Theorem 1

By our assumptions let

$$zf'(z)/f(z)=(1-a)p(z)+a, \tag{3.1}$$

then $p(z)$ is analytic in E satisfies $\mathrm{Re}(p(z))>0$ and is of the form

$$p(z)=1+c_mz^m+c_{m+1}z^{m+1}+\cdots, \tag{3.2}$$

for all z in E.

Differentiating (3.1) we have

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}+\frac{(1-a)zp'(z)}{(1-a)p(z)+a}=a+(1-a)p(z)+\frac{(1-a)zp'(z)}{(1-a)p(z)+a}. \tag{3.3}$$

Therefore $f(z)$ will be convex of order a, namely $1+\mathrm{Re}(zf''(z)/f(z))>b$, if

$$(a-b)+(1-a)p(z)+\frac{(1-a)zp'(z)}{(1-a)p(z)+a}>0. \tag{3.4}$$

From (2.3) and (2.4) with $a=0$ we know that (3.4) will hold provided

$$(a-b)+(1-a)\mathrm{Re}(p(z))\left[1-\frac{2mt/(1-t^2)}{a+(1-a)\{(1-t)/(1+t)\}}\right]>0,\quad |z|^m=t, \text{ namely}$$

$$(a-b)+(1-a)\mathrm{Re}(p(z))\left[\frac{T(t)}{(1-t)\{(2a-1)t+1\}}\right]>0, \tag{3.5}$$

where $T(t)=(1-2a)t^2+(2a-2m-2)t+1$.

Since $T(0)=1>0$, $T(1)=-2m<0$, therefore $T(t)=0$ has exactly one positive root between 0 and 1. Let r_1 be this positive root. Then $0<r_1<1$ and $T(t)>0$ for $0\leq t<r_1$.

Hence using (2.3) we see, provided $0\leq t<r_1$, that (3.5) will hold if

$$(a-b)+(1-a)\left(\frac{1-t}{1+t}\right)\left[\frac{T(t)}{(1-t)\{(2a-1)t+1\}}\right]>0, \tag{3.6}$$

i. e.,

$$\begin{aligned} Q(t)&\equiv(a-b)(1+t)[(2a-1)t+1]+(1-a)T(t)\equiv \\ &(4a^2-4a+1-2ab+b)t^2+(4a+2am-2ab-2m-2)t+(1-b)>0. \end{aligned} \tag{3.7}$$

Since $a\leq b$, therefore $Q(r_1)=(a-b)(1+r_1)[(2a-1)r_1+1]\leq 0$, also $Q(0)=1-b>0$, hence $Q(t)=0$ has a positive root between 0 and r_1. If r_0 is the smallest positive root of $Q(t)=0$, then $0<r_0\leq r_1$ and $Q(t)>0$ for $0\leq|z|^m<r_0$. The inequality (3.4) is thus seen to be satisfied if $z<r_0^{1/m}$, which means that $f(z)$ is starlike of order b in the region $|z|<r_0^{1/m}$.

To see that the result is sharp, let us consider the function $f(z)=z(1-z^m)^{(2a-2)/m}$. It is clearly that $f(z)$ belongs to $S(a)$ and

$$\frac{zf''(z)}{f'(z)}+1-b=\frac{T(-z^m)}{(1-z)^m[1+(1-2a)z^m]}.$$

Thus $\{zf''(z)/f'(z)\}+1-b=0$ for $z=(-r_0)^{1/m}$. Hence $f(z)$ is not convex of order b in any disk $|z|<r^{1/m}$ if $r>r_0$. This completes the the proof of Theorem 1.

For special case of this theorem when $a=b=0$, since $Q(t)=t^2-2(m+1)t+1$, therefore we have the following result which is well-known when $m=1$.

COROLLARY 1. *If $f(z)$ is starlike in E and is of the form $f(z)=z+a_{m+1}z^{m+1}+a_{m+2}z^{m+2}+\cdots$, $m\geq 1$, then $f(z)$ is convex in the region $|z|^m<m+1-(m^2+2m)^{\frac{1}{2}}$. This result is sharp.*

4. Proof of Theorem 2

Put $zf'(z)/f(z)=p(z)$, then we have, on differentiation that

$$1+\frac{zf''(z)}{f(z)}=\frac{zf'(z)}{f(z)}+\frac{zp'(z)}{p(z)}=p(z)+\frac{zp'(z)}{p(z)}. \tag{4.1}$$

Since $f(z)\in M(a)$, namely $\left|p(z)-\frac{1}{2a}\right|<\frac{1}{2a}$, therefore from (2.5), (2.7) and (4.1) we know that $f(z)$ is convex of order b, namely $1+\mathrm{Re}(zf''(z)/f'(z))>b$, if

$$\frac{1-t}{1+ct}-\frac{(1+c)mt}{(1-t)(1+ct)}>b, \tag{4.2}$$

or

(4.3) $$H(t) \equiv (1-t)^2 - (1+c)mt - b(1-t)(1+ct)$$
$$\equiv (1+bc)t^2 - (2+m+mc+bc-b)t + (1-b) > 0,$$

where $t = |z|^m$ and $c = 1-2a$.

Since $H(0) = 1-b > 0$, $H(1) = -(m+mc) < 0$, and $1+bc > 0$, therefore $H(t) = 0$ has two positive roots and the smallest of these two roots is between 0 and 1. Let R be this smallest root, then $H(t) > 0$, namely $f(t)$ is convex of order b, for $0 \leq |z|^m < R$, this completes the proof of Theorem 2.

For sharpness, let us consider the function

$$f(z) = \begin{cases} z \exp(z^m/m) & \text{for } c=0 \\ z\{1-cz^m\}^{-(1+c)/cm} & \text{for } c \neq 0. \end{cases}$$

It is easy to show that $f(z) \in M(a)$, and

$$1 + \frac{zf''(z)}{f(z)} - b = \begin{cases} [z^{2m} - (2+m-b)(-z^m) + (1-b)]/(1+z^m), & \text{for } c=0, \\ H(-z^m)/(1+z^m)(1-cz^m), & \text{for } c \neq 0. \end{cases}$$

This show that our function $f(z)$ is not convex of order b in any disk $|z| < R'$ if R' exceeds R.

Theorem 2 reduces to a result of R. Singh in [2] and [3] as a special case when $c=0$ and $m=1$.

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