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ON SOME MAPPINGS IN S-DISTRIBUTION THEORY

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1. Introduction

S-distribution theory is the discrete analogue of Schwartz's theory of distribution. This was first developed by Knoshaug [3] and was extended further by Sabharwal [6]. For other approaches to this theory, we mention the works of Traub [5], Moore [4], Berge [1], and Dector and Perry, Jr. [2] (on Gaussian integers). Let F be the space of all complex-valued functions defined on the set of all integers I. The product of two elements f,g in F is the convolution product and is denoted by $f*g$. In F, the subset of all functions with bounded support is denoted by B. The space of all continuous linear functionals on B is denoted by B', the space of S-distributions. E' denotes the space of S-distributions with bounded support.

DEFINITION 1.1. $f\in B'$ is said to be a *regular S-distribution* if

$$\langle f,g\rangle=\sum_{x\in I} f(x)\, g(x) \quad \text{for all } g \text{ in } B.$$

Let $\{e_n\}_{n=-\infty}^{\infty}$ be a sequence such that

$$\begin{aligned} e_n(x)&=1, \quad x=n \\ &=0, \quad x\neq n \end{aligned}$$

DEFINITION 1.2. A *k-regular S-distribution* $f^{\{k\}}$ is defined by the equation

$$\langle f^{\{k\}},\ g\rangle=f*g(k) \text{ for all } g \text{ in } B.$$

This $f^{\{k\}}$ is in fact given by a regular S-distribution $f(k-x)=f^{\{k\}}(x)$.

DEFINITION 1.3. A linear mapping η from B into B' is said to be *passive* if $\mathrm{Re}(\sum_{x\leq x_0} \overline{\eta g(x)}\, g(x))\geq 0$ for all $g\in B$ and for all $x_0\in I$.

DEFINITION 1.4. A linear mapping η from B into B' is said to be *causal* if $g(x)=0$ for $x\leq x_0$ implies that $\eta g(x)=0$ for $x\leq x_0$.

DEFINITION 1.5. A linear mapping η from B' into itself is said to be *translation-invariant* if it commutes with the shifting operator σ_k defined as follows:

$$\sigma_k f(x)=f(x-k).$$

2. Convolutional representation, and causality and passivity.

The following two theorems proved by Sabharwal [6] are made use of in the sequel.

THEOREM 2.1. *Let $f \in B'$. Then there exists a sequence $\{f_n\}$ of regular S-distributions in $B \cap B'$ converging in B' to f and hence f belongs to F.*

THEOREM 2.2. *Every S-distribution with bounded support can be written as a finite linear combination of k-regular S-distributions $e_0^{\{k\}}$.*

Now we prove two theorems in S-distribution theory (with less restrictions on the mappings) which are analogous to two theorems in Schwartz's distribution theory [7].

THEOREM 2.3. *A linear and translation-invariant mapping η from B' into itself is a convolution operator over E', that is, there exists a unique S-distribution ω in B' such that $\eta f = \omega * f$ for at least all f in E'.*

PROOF. Define ω as the distribution that η assigns to e_0. Thus $\eta e_0 \overset{\Delta}{=} \omega = \omega * e_0$ (as e_0 is the multiplicative identity; $e_0(x)=1$ if $x=0$ and zero otherwise). Since η commutes with the shifting operator

$$\sigma_k \eta e_0(x) = \omega(x-k) = \eta \sigma_k e_0(x) = \eta e_0(x-k)$$
$$= \omega(x) * e_0(k-x) = \omega(x) * e_0^{\{k\}}(x)$$
$$\{e_0(x) = e_0(-x)\}, \text{ i.e., } \eta e_0^{\{k\}}(x) = \omega(x) * e_0^{\{k\}}(x).$$

By Theorem 2.2 and linearity of η, we have

$$\eta f = \eta\Big(\sum_{p \le n \le q} \langle f, e_n \rangle e_n(x)\Big) = \eta\Big(\sum_{p \le n \le q} \langle f, e_n \rangle e_0^{\{n\}}(x)\Big)$$
$$= \sum_{p \le n \le q} \langle f, e_n \rangle \eta e_0^{\{n\}}(x) = \omega(x) * \sum_{p \le n \le q} \langle f, e_n \rangle e_0^{\{n\}}(x) = \omega * f$$

where $f \in E'$. For uniqueness, assume the existence of two distributions ω_1 and ω_2 such that $\eta f = \omega_1 * f = \omega_2 * f$ for every f in E'. Then for each φ in B. we may write $\langle \omega_1(k), \varphi(x-k) \rangle = \langle \omega_2(k), \varphi(x-k) \rangle$ as $B \subset E'$. As $\varphi(k)$ traverses B, $\varphi(x-k)$ also traverses B for any fixed value of x. Hence $\omega_1 = \omega_2$.

THEOREM 2.4. *If a linear mapping from B into B' is passive, then it is also causal.*

PROOF. Let f and f_1 be two members of B and let $v=\eta f$ and $v_1=\eta f_1$. Assume that $f(x)=0$ for all $x\leq x_0$. We shall show that $v(x)=0$ for all $x\leq x_0$ too. Let α be an arbitrary complex number and let $f_2=f_1+\alpha f$. Then $f_2(x)=f_1(x)$ for $x\leq x_0$. Moreover if $v_2 \overset{\Delta}{=} v_1+\alpha v$, then by linearity of η, $v_2=\eta f_2$. Also, by the passivity of η and by Theorem 2.1

$$\mathrm{Re}\,\Big(\sum_{x\leq x_0} \overline{v_2(x)} f_2(x)\Big)\geq 0, \quad \text{i.e.,}$$

$$\mathrm{Re}\,\Big(\sum_{x\leq x_0} \overline{v_1(x)} f_1(x)\Big)+\mathrm{Re}\{\overline{\alpha}\,\Big(\sum_{x\leq x_0} \overline{v(x)} f_1(x)\}\geq 0 \qquad (*)$$

Since the inequality (*) must hold for all complex α, the second summation, i.e., $\sum_{x\leq x_0} \overline{v(x)} f_1(x)$ must be zero for all $x\leq x_0$. This implies that $v(x)f_1(x)=0$ for all $x\leq x_0$. Since $f_1(x)$ is an arbitrary element in B, it follows that $v(x)=0$ for all $x\leq x_0$.

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