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ON A FUNCTIONAL INEQUALITY

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1. In this paper we consider the functional inequality

(A) $$f[\theta(x,y)] \leq \Psi[f(x),f(y)]$$

which is a generalization of Jensen's inequality, namely $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$. With suitable assumptions on θ and Ψ we are able to generalize some of the "boundedness implies continuity" results such as these of Jensen [3] and Berstein [2]. Throughout this paper X is a topological space, $\theta: X\times X \to X$ and $\Psi: R^2 \to R$ where R denotes the set of real numbers. If $T \subset X$ let $\theta_1(T)=T$ and $\theta_{n+1}(T)=\theta[\theta_n(T), \theta_n(T)]$ for $n=1, 2, 3, \cdots$. For $x_0 \in X$ define $[x_0]_\theta^1 = \{\theta(x_0, x) \mid x \in X\}$, $[x_0]_\theta^2 = \theta([x_0]_\theta^1, X)$, $\cdots$, $[x_0]_\theta^{n+1} = \theta([x_0]_\theta^n, X)$ for $n=1, 2, 3, \cdots$.

We make the following assumptions on θ and Ψ throughout this paper.

(θ.1) Reflexivity: $\theta(x,x)=x$ for all $x \in X$.

(θ.2) For each $x_0 \in X$, $y \to \theta(x_0, y)$ is a one-to-one mapping of X into X.

(θ.3) For each $x_0 \in X$, $\bigcup_{n=1}^{\infty} [x_0]_\theta^n = X$.

(θ.4) For each $x_0 \in X$, there exists a neighbourhood U of x_0 such that $\bigcap_{x\in U} \theta(x, X)$ is a neighbourhood of x_0.

(θ.5) If we define $\theta^{-1}(x,z)=y$ whenever $x,y,z \in X$ and $\theta(x,y)=z$, then θ^{-1} is continuous on its domain, i.e. $(\lim_{\gamma,\delta})\theta^{-1}(x_\gamma, y_\delta) = \theta^{-1}(x_0, y_0)$ whenever $x_\gamma \to x_0$, $y_\delta \to y_0$ and (x_γ, y_δ), (x_0, y_0) are in the domain of θ^{-1}.

(θ.6) For any $(x_0, y_0) \in X \times X$ there exists a neighbourhood U_{x_0} of x_0 such that $\bigcup_{n=1}^{\infty} \theta_n[\theta(U, y_0)]$ is a neighbourhood of $\theta(x_0, y_0)$ for any open neighbourhood U of x_0 contained in U_{x_0}.

(Ψ.1) Ψ is continuous in the first variable.

(Ψ.2) Ψ is monotonic increasing and upper semicontinuous in the second variable.

(Ψ.3) $\lim_{s \to -\infty} \Psi(s, t) = -\infty$ (Hence define $\Psi(-\infty, t) = -\infty$) for any $t \in R$.

(Ψ.4) $\Psi(s,t)<\mathrm{Max}(s,t)$ if $s\neq t$.

Several examples of such functions will be presented later on.

A real-valued function on a topological space X is called *locally bounded above* if f is bounded above on some neighbourhood of each point of X.

2. THEOREM 1. *Suppose f is a real-valued function on the topological space X and satisfies inequality (A) for all $x,y\in X$ where θ satisfies (θ.1), (θ.2), (θ.3), (θ.4), (θ.5) and (θ.6) and Ψ satisfies (Ψ.1), (Ψ.2), (Ψ.3)* and *(Ψ.4). If f is bounded above on a subset T of X such that $\bigcup_{n=1}^{\infty}\theta_n(T)$ contains a nonempty open subset of X, then f is continuous.*

The proof of the theorem is divided into four lemmas.

LEMMA 1. *If $f(t)\leq K$ for all $t\in T$, then $f(x)\leq K$ for all $x\in\bigcup_{n=1}^{\infty}\theta_n(T)$.*

It is easy to prove Lemma 1 by induction.

LEMMA 2. *Suppose $\lim_{k\to\infty}u_k=u$ $(-\infty\leq u<\infty)$ and $\limsup_{n\to\infty}v_n=v(-\infty<v<\infty)$. Then*

$$\limsup_{k,n\to\infty}\Psi(u_k,v_n)\leq\Psi(u,v).$$

PROOF. Given $\varepsilon>0$, there exists a natural number N such that $v_n\leq v+\varepsilon$ whenever $n\geq N$. By (Ψ.2) we have

$$\Psi(u_k,v_n)\leq\Psi(u_k,v+\varepsilon) \text{ if } n\geq N. \text{ Hence}$$

$$\limsup_{k\to\infty}\Psi(u_k,v_n)\leq\limsup_{k\to\infty}\Psi(u_k,v+\varepsilon).$$

If $u=-\infty$, then we have, by (Ψ.3)

$$\limsup_{k,n\to\infty}\Psi(u_k,v_n)\leq\limsup_{k\to\infty}\Psi(u_k,v+\varepsilon)=-\infty.$$

If $u>-\infty$, then we have by (Ψ.1),

$$\limsup_{k\to\infty}\Psi(u_k,v+\varepsilon)=\Psi(u,v+\varepsilon) \text{ and by } (\Psi.2)$$

$$\limsup_{k,n\to\infty}\Psi(u_k,v_n)\leq\Psi(u,v).$$

LEMMA 3. *If $f(x)\leq K$ for all x belonging to some neighbourhood U_{x_0} of x_0, then f is continuous at x_0.*

PROOF. Let $\limsup_{x\to x_0}f(x)=M$ and $\liminf_{x\to x_0}f(x)=m$. Choose two nets $\{x_\gamma\}$ and $\{z_\delta\}$ in X such that $x_\gamma\to x_0$, $z_\delta\to x_0$, $\lim_\gamma f(x_\gamma)=M$ and $\lim_\delta f(z_\delta)=m$. By ($\theta$.4),

$y_{(\gamma,\delta)}=\theta^{-1}(x_\gamma, z_\delta)$ is defined whenever x_γ and z_δ are sufficiently close to x_0. By $(\theta.5)$ we have $\lim_{(\gamma,\delta)} y_{(\gamma,\delta)}=x_0$. Hence

$$f(z)\leq\Psi(f(x),\ f[y_{(\gamma,\delta)}]).$$

Suppose $m<M$. Then, by Lemma 2, we have

$$M\leq\Psi(m,\ \limsup_{(\gamma,\delta)} f[y_{(\gamma,\delta)}])\leq\Psi(m,\ M)<M.$$

This is a contradiction. Hence $m=M$.

LEMMA 4. *If f is bounded above on some neighbourhood U_{x_0} of x_0, then f is locally bounded above on X.*

PROOF. Suppose $f(x)\leq K$ for all $x\in U_{x_0}$. If $z_0\in[x_0]^1_\theta$, then, by $(\theta.2)$, there exists a unique $y_0\in X$ such that $\theta(x_0, y_0)=z_0$. By $(\theta.6)$ we can find a neighbourhood U of x_0, $U\subset U_{x_0}$, such that $\bigcup_{n=1}^{\infty}\theta_n[\theta(U, y_0)]$ is a neighbourhood of z_0. By Lemma 1, f is bounded above by $\mathrm{Max}(K, f(y_0))$ on $\bigcup_{n=1}^{\infty}\theta_n[\theta\ (U, y)]$ which is a neighbourhood of z_0. Suppose that $n\geq 1$, f is bounded above in some neighbourhood of each point of $[x_0]^n_\theta$. If $v\in[x_0]^{n+1}_\theta$, then $v=\theta(a, b)$ where $a\in[x_0]^n_\theta$ and $b\in X$. By $(\theta.6)$ there exists a neighbourhood U of a such that $\bigcup_{n=1}^{\infty}\theta_n[\theta(U,\ b)]$ is a neighbourhood of v and f is bounded above on U by some constant, say M. By the same procedure, f is bounded above on $\bigcup_{n=1}^{\infty}\theta_n[\theta(U, b)]$ by $\mathrm{Max}(M,\ f(b))$. Hence the lemma is proved.

3. Examples

(a) Suppose σ is a homeomorphism of a topological space Y onto the topological space X. Then it is easy to show that the function $\theta_\sigma: Y\times Y\to Y$ defined for x, $y\in Y$ by $\theta_\sigma(x, y)=\sigma^{-1}[\theta(\sigma(x), \sigma(y))]$ satisfies the same conditions $(\theta.1)$, $(\theta.2)$, $\cdots$ and $(\theta.6)$ on $Y\times Y$ as θ does on $X\times X$. In particular, if X admits at least one θ and has a rich collection of homeomorphisms, X will admit several such θ's.

(b) Suppose Δ is an open, convex subset of $R^N (N\geq 1)$ and $0<\lambda<1$. Define $\theta: \Delta\times\Delta\to\Delta$ by $\theta(x, y)=\lambda x+(1-\lambda)y$ for all $x, y\in\Delta$. Then it is easy to see that θ satisfies $(\theta.1)$, $(\theta.2)$, $\cdots$ and $(\theta.6)$. In particular, if $\Delta=\{x|a<x<b,\ x \text{ a real number}\}$, then the quasilinear mean ([1], page 240—241),

$$\theta(x, y)=\sigma[\lambda\sigma^{-1}(x)+(1-\lambda)\sigma^{-1}(y)],$$

where σ is a continuous strictly monotonic function from Δ onto Δ, satisfies $(\theta.1)$, $(\theta.2)$, $\cdots$ and $(\theta.6)$. Typical examples of quasilinear means on $R^+=(0, \infty)$ are:

$$(x,y)\longrightarrow \sqrt[r+s]{x^r y^s}\ ,\ (r,s>0 \text{ but fixed}),$$

$$(x,y)\longrightarrow \frac{xy}{\lambda x+(1-\lambda)y}\ ,\ (0<\lambda<1 \text{ but fixed})$$

$$(x,y)\longrightarrow \sqrt[m]{\lambda x^m+(1-\lambda)y^m}\quad (0<\lambda<1 \text{ fixed; } m \text{ a fixed natural number}).$$

(c) Let $\varphi: R\rightarrow R$, be continuous on R with continuous first derivative φ' and satisfies the conditions:

$$\varphi'(x)\leq 1 \ (\text{for all } x\in R)$$
$$\varphi(x)<kx \ (\text{fixed } k>0 \text{ and for all } x<0)$$
$$\varphi(x)<x \ (\text{for all } x>0).$$

Then clearly $\Psi(x,y)=\varphi(x-y)+y$ is a mapping on R^2 into R and satisfies $(\Psi.1)$, $(\Psi.2)$, $(\Psi.3)$ and $(\Psi.4)$.

The following corollaries are easy consequence of Theorem 1 and Theorem of Steinhaus [8].

COROLLARY 1 [6]. *Suppose f is a convex function defined on the real interval (a,b). If f is bounded above on a subset T such that the inner Lebesgue measure of the mid-point convex hull of T (see* [6], [7]*) is positive, then f is continuous.*

COROLLARY 2. *Suppose f satisfies the inequality*

$$f(\sum_{i=1}^{n}\alpha_i x_i)\leq \sum_{i=1}^{n}\beta_i f(x_i)$$

for all real x_1, x_2, $\cdots$, x_n where α_1, α_2, $\cdots$, α_n, β_1, β_2, $\cdots\beta_n$ are fixed $\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\beta_i=1$ and $\alpha_i, \beta_i>0$ $(1\leq i\leq n)$.
If f is bounded above on a set T of positive Lebesgue measure, then f is continuous.

COROLLARY 3([5] Theorem 1, [4]). *Suppose a root function γ_2 (see* [4]*) is defined and continuous on an additive abelian locally compact (or Baire) group G and f is a convex function on G (see* [4]*). If f is bounded above on a subset T of G such that*

$$J(T)=\bigcup_{n=0}^{\infty}\gamma_2^n(\sum_{1}^{2^n} T)$$

contains a set of positive finite Haar measure (or $J(T)$ contains a nonmeager almost open subset of G), then f is continuous.

PROOF. Define $\theta(x,y)=\gamma_2(x+y)$ for all $x,y\in G$. Then it is easy to see that θ satisfies the conditions of Theorem 1 and $J(T)=\bigcup_{n=1}^{\infty}\theta_n(T)$. By using Steinhaus theorem (resp. Corollary 3.1 [4]) and Theorem 1, f is continuous on G.

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