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GEOMETRICAL STUDY OF A NEW CURVATURE TENSOR

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1. Introduction

We consider an n-dimensional real differentiable manifold M_n of differentiability class C^{r+1}. Let there exists a vector valued real linear function F such that for arbitrary vector fields X, Y, Z, etc

(1.1) $$\overline{\overline{X}}=X$$

where

(1.2) $$\overline{X}\overset{\text{def}}{=\!=}F(X).$$

Let us further suppose that in M_n there be a positive definite Riemannian metric tensor g, such that

(1.3) $$g(\overline{X},\overline{Y})=g(X,Y),$$

then M_n is called an almost product manifold.

Let us put

(1.4) $$'F(X,Y)\overset{\text{def}}{=\!=}g(\overline{X},Y).$$

Then from (1.1), (1.2), (1.3) and (1.4), we have

(1.5)a $$'F(X,Y)=g(\overline{X},Y)=g(X,\overline{Y})='F(Y,X)$$

(1.5)b $$'F(\overline{X},\overline{Y})=g(X,\overline{Y})='F(X,Y)$$

and

(1.5)c $$'F(\overline{X},Y)=g(X,Y)='F(X,\overline{Y}).$$

Let D be a Riemannian connexion in M_n,

then

(1.6)a $$D_XY-D_YX=[X,Y]$$

(1.6)b $$(D_Xg)\ (Y,Z)=0$$

If in addition to (1.1), (1.3) and (1.6) the condition

(1.7) $$(D_XF)(Y)=0$$

is satisfied, we say that M_n is almost product and almost decomposable manifold.

If we put

(1.8) $$'R(X,Y,Z,T)\overset{\text{def}}{=\!=}g(R(X,Y,Z),\ T)$$

then in an almost product and almost decomposable manifold, we have

(1.9)a $$'R(X,Y,\overline{Z},T)='R(X,Y,Z,\overline{T})='R(\overline{X},\overline{Y},\overline{Z},T)$$

(1.9)b $'R(X,Y,\overline{Z},\overline{T})='R(X,Y,Z,T)='R(\overline{X},\overline{Y},\overline{Z},\overline{T}).$

Let Ric be Ricci tensor of M_n then for an almost product and almost decomposable manifold, we have

(1.10)a $\mathrm{Ric}(\overline{Y},Z)=\mathrm{Ric}(Y,\overline{Z})$

(1.10)b $\mathrm{Ric}(\overline{Y},\overline{Z})=\mathrm{Ric}(Y,Z).$

In a recent paper the author [1] has defined a new curvature tensor to explore its relativistic significance

(1.11) $W_3(X,Y,Z)\overset{\text{def}}{=\!=}R(X,Y,Z)+\frac{1}{n-1}[g(Y,Z)\gamma(X)-Y\mathrm{Ric}(X,Z)]$

where Ric $(X,Z)\overset{\text{def}}{=\!=}g(\gamma(X),Z).$

If we put

(1.12) $'W_3(X,Y,Z,T)\overset{\text{def}}{=\!=}g(W_3(X,Y,Z),T)$

then we get

(1.13) $'W_3(X,Y,Z,T)='R(X,Y,Z,T)+\frac{1}{n-1}[g(Y,Z)\mathrm{Ric}(X,T)-g(Y,T)\mathrm{Ric}(X,Z)].$

Further we have

(1.14) $'W_3(X,Y,Z,T)=-'W_3(X,Y,T,Z)$

and

(1.15) $'W_3(X,Y,Z,T)+'W_3(X,Z,T,Y)+'W_3(X,T,Y,Z)=0.$

2. In this section we shall study the properties of $'W_3(X,Y,Z,T)$ in an almost product and almost decomposable manifold M_n.

THEOREM 2.1. *In an almost product and almost decomposable manifold, we have*

(2.1)a $'W_3(\overline{X},Y,Z,\overline{T})='W_3(X,\overline{Y},\overline{Z},T)$

(2.1)b $'W_3(\overline{X},\overline{Y},\overline{Z},T)='W_3(X,Y,Z,\overline{T})$

(2.1)c $'W_3(\overline{X},\overline{Y},Z,T)='W_3(X,Y,\overline{Z},\overline{T})$

and

(2.1)d $'W_3(\overline{X},\overline{Y},\overline{Z},\overline{T})='W_3(X,Y,Z,T).$

PROOF. Barring the vectors in (1.13), we get

(2.2) $'W_3(\overline{X},Y,Z,\overline{T})='R(\overline{X},Y,Z,\overline{T})+\frac{1}{n-1}[g(Y,Z)\ \mathrm{Ric}\ (X,T)-g(Y,\overline{T})\ \mathrm{Ric}\ (\overline{X},Z)]$

and

(2.3) $\quad 'W_3(X,\overline{Y},\overline{Z},T)='R(X,\overline{Y},\overline{Z},T)+\frac{1}{n-1}[g(\overline{Y},\overline{Z}) \text{ Ric } (X,T)$
$-g(\overline{Y},T) \text{ Ric } (X,\overline{Z})]$.

Using (1.3), (1.5)a (1.10)b and (1.9)b (after barring Y and Z) in the above equations and comparing them, we get (2.1)a. Barring the vectors Y, Z and T in (2.1)a and using (1.1), we get (2.1)b. Similarly the other two results can be obtained by barring the vectors.

THEOREM 2.2. *In an almost product and almost decomposable manifold, we have*

(2.4)a $\quad 'W_3(\overline{X},\overline{Y},Z,T)+'W_3(\overline{X},\overline{Z},T,Y)+'W_3(\overline{X},\overline{T},Y,Z)=0.$

(2.4)b $\quad 'W_3(X,Y,\overline{Z},\overline{T})+'W_3(X,Z,\overline{T},\overline{Y})+'W_3(X,T,\overline{Y},Z)=0.$

PROOF. Using (1.5)a, (1.9)b, (1.10)a and (1.13), we get the result.

DEFINITION 2.1. We shall call a manifold to be *W_3-flat* if

(2.5) $\quad 'R(X,Y,Z,T)=\frac{1}{n-1}[g(Y,T) \text{ Ric}(X,Z)-g(Y,Z) \text{ Ric}(X,T)]$

THEOREM 2.3. *When $\overline{X}\neq\pm X$, a W_3-flat almost product and almost decomposable manifold is flat.*

PROOF. From (2.5), we see that if the almost product and almost decomposable manifold be W_3-flat, we have

(2.6) $\quad (n-1)\ 'R(X,Y,Z,T)=g(Y,T) \text{ Ric}(X,Z)-g(Y,Z) \text{ Ric}(X,T)$

Barring the vectors Z, T in this equation and using (1.5)a and (1.9), we get

(2.7) $\quad (n-1)\ 'R(X,Y,Z,T)='F(Y,T) \text{ Ric}(X,\overline{Z})-'F(Y,Z) \text{ Ric}(X,\overline{T})$

Now barring Z and then T in (2.6) and using (1.5)a and (1.9)a, we obtain

(2.8) $\quad (n-1)\ 'R(X,Y,\overline{Z},T)=g(Y,T) \text{ Ric}(X,\overline{Z})-'F(Y,Z) \text{ Ric}(X,T)$

and

(2.9) $\quad (n-1)\ 'R(X,Y,Z,\overline{T})='F(Y,T) \text{ Ric}(X,Z)-g(Y,Z) \text{ Ric}(X,\overline{T})$.

These equations hold for arbitrary X,Y,Z,T. Since $g(X,T)\neq 'F(X,T)$ and $\text{Ric}(Y,Z)\neq\text{Ric}(Y,\overline{Z})$, the right hand sides and consequently the left hand sides of these equations must vanish. Hence we have the statement.

3. In this section we shall study some properties of W_3-curvature tensor in almost Tachibana space and almost Kähler space.

Here we consider an even dimensional real manifold M_{2n} differentiability class C^{r+1}. Let F be the vector valued linear function defined in M_{2n} such that

(3.1) $$\overline{\overline{X}}=-X, \text{ where } \overline{X}\overset{\text{def}}{=\!=}F(X)$$

then M_{2n} is called an almost complex manifold.

If the metric tensor g:

(3.2) $$g(\overline{X},\overline{Y})=g(X,Y)$$

be defined in M_{2n}, then it is called an almost Hermite manifold.

Let D be Riemannian connexion in M_{2n}:

(3.3)a $$D_XY-D_YX=[X,Y]$$

(3.3)b $$D_Xg(Y,Z)=g(D_XY,Z)+g(Y,D_XZ).$$

In M_{2n}, a tensor,

(3.4) $$A(X,Y,Z)=(D_XA)(Y,Z)+(D_YA)(Z,X)+(D_ZA)(X,Y)$$

be defined such that it is skew symmetric in X,Y and Z where $A(X,Y)$ is a skew symmetric tensor.

If in an almost Hermite manifold, the tensor $A(X,Y,Z)$ vanishes identically, the manifold is called almost Kähler manifold.

Now we put

(3.5) $$G(X,Y)=(D_XF)(Y)+(D_YF)(X).$$

If in an almost Hermite manifold, the tensor $G(X,Y)$ vanishes identically, the manifold is called almost Tachibana manifold.

We now consider the almost Kähler manifold M_{2n}. Suppose our almost Kähler manifold is W_3-flat, then we write (2.5) as

(3.6) $$R_{hijk}=\frac{1}{n-1}(g_{ik}R_{hj}-g_{ij}R_{hk})$$

Transvecting this equation by $F^{hi}F^{jk}$

(3.7) $$R=-(n-1)H$$

where $H_{hi}=\frac{1}{2}R_{hijk}F^{jk}$ and $H=-F^{ij}H_{ij}$.

From (3.7) we have

(3.8) $$R-H=\left(\frac{n}{n-1}\right)R.$$

It is well known [2] that in an almost Kähler manifold, we have

(3.9) $$R\leq H$$

and in an almost Tachibana manifold, we have

(3.10) $$R\leq H$$

The equality sign occurs when and only when the manifold is a Kähler manifold.

From (3.8) and (3.9) we have the following theorem.

THEOREM 3.2. *If an almost Kähler manifold is W_3-flat, then the scalar curvature R satisfies*

(3.11) $$R \leq 0$$

and $R=0$ if and only if the manifold is a Kähler manifold.

Similarly from (3.8) and (3.10), we have

THEOREM 3.3. *If an almost Tachibana manifold is W_3-flat, then the scalar curvature R satisfies*

(3.12) $$R \leq 0$$

and $R=0$ if and only if the manifold is a Kähler manifold.

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REFERENCES

[1] G. P. Pokhariyal, *Curvature tensors and their relativistics significance III*, Yorkohama Math. Jour. Vol. XXI, No. 2 pp. 115—119, (1973)

[2] K. Yano, *Differential Geometry on Complex spaces*. Pergamon Press, pp. 181—197(1964)