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ON H-PROJECTIVE RECURRENT KÄHLER SPACES

By Ram Hit

1. Introduction

A real $2n$-dimensional space M_{2n} with a Riemannian metric g_{ji} is a Kähler space if there exists a mixed tensor $F_j^{\ i}$ which satisfies

(1.1)
$$F_j^{\ i}F_i^{\ h}=-\delta_j^{\ h},\ F_j^{\ t}F_i^{\ s}g_{ts}=g_{ji},$$
$$\nabla_j F_i^{\ h}=0,$$

∇_j being the operator of covariant differentiation with respect to Riemannian connexion $\{\ \}$ formed with g_{ji}. Let us put

(1.2)
$$F_{ji}=g_{ri}F_j^{\ r}$$

Then from (1.1) and (1.2) we have

(1.3)
$$F_{ji}=-F_{ij},\ F^{ji}=g^{jr}F_r^{\ i}=-F^{ij}.$$

Let R^h_{kji} be the curvature tensor with respect to the Riemannian connexion $\{\ \}$ and $R_{ji}=R^h_{hji}$ be the Ricci tensor. Then in a Kähler space the following hold (Yano, 1965)

(1.4) a) $H_{ji}=-H_{ij}$, b) $R_{ks}F_j^{\ s}=H_{kj}$,

c) $H_{ks}F^s_{\ j}=-R_{kj}$, d) $H_{kj}F^{kj}=-R$,

where

$$R_{kjih}=R^r_{kji}g_{rh},\ R=R_{ji}g^{ji} \text{ and } H_{ji}=\frac{1}{2}R_{ijkh}F^{kh}.$$

2. H-projective recurrent Kähler space

It is well known that H-projective curvature tensor P^h_{ijk} may be considered as the tensor corresponding to projective curvature tensor W^h_{ijk}. It can be easily proved that a projective recurrent Riemannian space is a recurrent space iff $\nabla_l R=A_l R$ where A_l is the vector of recurrence. Now, let us consider the projective recurrent Kähler space, that is,

(2.1) $$W_{ijkh}=A_l W_{ijkh}.$$

We can therefore write (2.1) as

(2.2) $$\nabla_l R_{ijkh}+\frac{1}{2n-1}[g_{jh}\nabla_l R_{ik}-g_{ih}\nabla_l R_{jk}]$$
$$=A_l\left[R_{ijkh}+\frac{1}{2n-1}\{g_{jh}R_{ik}-g_{ih}R_{jk}\}\right].$$

Transvecting the above equation with F^{kh}, we obtain

(2.3) $$\nabla_l H_{ij}=A_l H_{ij},$$

which gives

(2.4) $$\nabla_l R_{ij}=A_l R_{ij}.$$

Consequently, we have

(2.5) $$\nabla_l R_{ijkh}=A_l R_{ijkh}.$$

Thus we have

THEOREM 2.1. *A projective recurrent Kähler space is a recurrent space.*

COROLLARY 2.1. *A projective symmetric Kähler space is of constant scalar curvature.*

A Kähler space is called H-projective recurrent (Mishra, 1970) if

(2.7) a) $$\nabla_l P^h_{ijk}=\lambda_l P^h_{ijk},$$

where λ_l is the vector of recurrence and P^h_{ijk} is the H-projective curvature tensor. From (2.7) a) we have

(2.7) b) $$\nabla_l P_{ijkh}=\lambda_l P_{ijkh},$$

where $P_{ijkh}=P^r_{ijk}g_{rh}$. Hence

(2.8) $$P_{ijkh}=R_{ijkh}-\frac{1}{2(n+1)}[g_{ih}R_{jk}-g_{jh}R_{ik}-F_{ih}H_{jk}+F_{jh}H_{ik}+2F_{kh}H_{ij}].$$

We can therefore write (2.7) b), as

(2.9) $$\nabla_l R_{ijkh}-\frac{1}{2(n+1)}[g_{ih}\nabla_l R_{jk}-g_{jh}\nabla_l R_{ik}-F_{ih}\nabla_l H_{jk}+F_{jh}\nabla_l H_{ik}+2F_{kh}\nabla_l H_{ij}]$$
$$=\lambda_l\left[R_{ijkh}-\frac{1}{2(n+1)}\{g_{ih}R_{jk}-g_{jh}R_{ik}-F_{ih}H_{jk}+F_{jh}H_{ik}+2F_{kh}H_{ij}\}\right].$$

Transvecting (2.9) with F^{ij} we get

(2.10) $$2\nabla_l H_{kh}-\frac{1}{n+1}\{2\nabla_l H_{kh}-\nabla_l R F_{kh}\}=\lambda_l\left[2H_{kh}-\frac{1}{n+1}\{2H_{kh}-RF_{kh}\}\right],$$

which gives $\nabla_l R_{jk}=\lambda_l R_{jk}$ iff $\nabla_l R=\lambda_l R$. Consequently, we have $\nabla_l R_{ijkh}=\lambda_l R_{ijkh}$.

Therefore we have

THEOREM 2.2. *An H-projective recurrent Kähler space is a recurrent space iff* $\nabla_l R=\lambda_l R$.

If v^h is an analytic H-projective vector which is not affine, we have

$$\underset{v}{\mathscr{L}}\, P^h_{kji}=0 \text{ and } \underset{v}{\mathscr{L}}\, \nabla_l P^h_{kji}=(\underset{v}{\mathscr{L}}\, \lambda_l) P^h_{kji}, \tag{2.11}$$

where $\mathscr{L}$ denotes the operator of Lie-derivative. By means of the well-known formula

$$\underset{v}{\mathscr{L}}\nabla_l P^h_{kji}-\nabla_l \underset{v}{\mathscr{L}} P^h_{kji}=P^r_{kji}\underset{v}{\mathscr{L}}\left\{{h \atop l\ r}\right\} - P^h_{kjs}\underset{v}{\mathscr{L}}\left\{{s \atop l\ i}\right\}-P^h_{kri}\underset{v}{\mathscr{L}}\left\{{r \atop l\ j}\right\}-P^h_{tji}\underset{v}{\mathscr{L}}\left\{{t \atop l\ k}\right\}. \tag{2.12}$$

In view of (2.11) we find from (2.12)

$$(\underset{v}{\mathscr{L}}\lambda_l)P^h_{kji}-\delta^h_l P^r_{kji}\,\rho_r+2\rho_l\, P^h_{kji}+\rho_k\, P^h_{lji}+\rho_j\, P^h_{kli}+\rho_i\, P^h_{kjl} + F^h_l\, P^r_{kji} F_r{}^s\,\rho_s-F^s_l\rho_r(F^r_k\, P^h_{sji}+F^r_j\, P^h_{ksi}+F_i{}^r\, P^h_{kjs})=0.$$

Contracting with respect to h and l, we get

$$(\underset{v}{\mathscr{L}}\lambda_l-2(n-1)\,\rho_l)\; P^l_{kji}=0. \tag{2.13}$$

If the recurrence vector λ_ρ is Lie-invariant then $P^r_{kji}\,\rho_r=0$. We have also from (2.11), (2.12) and the fact that λ_ρ is Lie-invariant.

$$0=P^r_{kji}\,\underset{v}{\mathscr{L}}\left\{{h \atop l\ r}\right\}-P^h_{kjs}\,\underset{v}{\mathscr{L}}\left\{{s \atop l\ i}\right\}-P^h_{kti}\,\underset{v}{\mathscr{L}}\left\{{t \atop l\ j}\right\}-P^h_{rji}\,\underset{v}{\mathscr{L}}\left\{{r \atop l\ k}\right\}. \tag{2.14}$$

Simplifying the right hand side of the above equation, we obtain

$$0=(\rho_l P_{hkji})(\rho^l P^{kjih})+2(\rho^r P_{rjih})(\rho_s P^{sjih})+(\rho^r P_{hjrs})(\rho_t P^{hjts})$$

where ρ_i is the associated vector of v^h. Therefore we have $P^h_{kji}\,\rho_h=0$. Consequently, we have

THEOREM 2.4. *If in an H-projective recurrent Kähler space an analytic H-projective vector, which is not affine leaves the recurrence vector λ_l, Lie-invariant, then the space is of constant holomorphic sectional curvature.*

In an H-projective recurrent Kähler space if v^h is an affine vector then from (2.9) we have $\underset{v}{\mathscr{L}}\,\lambda_l\, P^h_{kji}=0$. Thus if $\underset{v}{\mathscr{L}}\lambda_l\neq 0$, we get $P^h_{kji}=0$. Thus we have

THEOREM 2.5. *If in an H-projective recurrent Kähler space an analytic H-projective vector which is affine and* $\underset{v}{\mathscr{L}}\lambda_l \neq 0$, *then the space is H-projectively flat.*

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Banaras Hindu University
Varanasi-221005, India

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