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WEAKLY HAUSDORFF SPACES

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1. Introduction

The concept of weakly Hausdorff (hereafter denoted by w.h.) was defined by Levine in [4] in terms of nets and proved to be weaker than both the Hausdorff property and regularity. It is the purpose of this paper to explore more fully this concept.

In (2) we develop characterizations of w.h. spaces and prove that a space is w.h. iff its T_0-identification is T_2.

In (3) we consider subspaces, products, and images of w.h. spaces and compare the concepts of w.h., T_2, and regularity.

Theorem 4.2 gives a complete characterization of a finite w.h. topology while Theorem 4.7 concerns the structure of an infinite w.h. topology.

In a compact or paracompact space, the w.h. property and regularity are shown to coincide in (5). Moreover, we prove that closures of compact subsets of a w. h. space are compact.

In (6) we consider three different definitions of local compactness which occur in the literature and show they agree in w.h. spaces.

In (7) we prove the analogue of the standard result that a space is $T_{3.5}$ iff it is homeomorphic to a subspace of a compact, T_2 space—that is, we show a space is completely regular iff it is homeomorphic to a subspace of compact, w.h. space.

2. Characterizations

DEFINITION 2.1. A space $(X, \mathscr{T})$ is *w.h.* iff $c(x)=c(y)$ whenever there is a net $S: D \to X$ such that $\lim S=x$ and $\lim S=y$.

THEOREM 2.2. *$(X, \mathscr{T})$ is w.h. iff for each x, y in X one of the following holds:*

(a) $x \in O \in \mathscr{T}$ *iff* $y \in O \in \mathscr{T}$

(b) *There exist* $O, U \in \mathscr{T}$ *such that* $x \in O$, $y \in U$, *and* $O \cap U = \phi$.

PROOF. Necessity. Suppose $(X, \mathscr{T})$ is w.h. and for x, y in X condition

(b) does not hold. Then we can define a directed set $D=\{O\cap U : x\in O\in\mathcal{T}$ and $y\in U\in\mathcal{T}\}$ with ordering by reverse inclusion, and a net $S: D\to X$ by $S(O\cap U)\in O\cap U$, arbitrary. Then lim $S=x$ and lim $S=y$ and thus $c(x)=c(y)$, which is (a).

Sufficiency. Suppose lim $S=x$ and lim $S=y$ for some net $S: D\to X$. If $c(x)\neq c(y)$, then (a) fails and so by (b) there are disjoint, open sets O and U such that $x\in O$ and $y\in O$. But then S is eventually in both O and U, contradicting disjointness. Thus $c(x)=c(y)$, and $(X,\mathcal{T})$ is w.h.

DEFINITION 2.3. $(X,\mathcal{T})$ is an R_0 *space iff* $c(x)\subset O$ whenever $x\in O$ and O is open.

DEFINITION 2.4. $(X,\mathcal{T})$ is an R_1 *space* iff for $x,y\in X$, $c(x)$ and $c(y)$ either coincide or are contained in disjoint open sets.

REMARK 2.5. The two previous definitions were introduced by Davis in [1], and after the next lemma we shall prove that R_1 and w.h. are identical properties.

LEMMA 2.6. *A w.h. space is* R_0.

PROOF. Suppose $x\in O$ where O is open and let $y\in c(x)$. Then condition (b) of Theorem 2.2 fails and so, by (a), $y\in O$ since $x\in O$. Thus $c(x)\subset O$ and the space is R_0.

THEOREM 2.7. $(X,\mathcal{T})$ *is w.h. iff* $(X,\mathcal{T})$ *is* R_1.

PROOF. Necessity. Suppose $(X,\mathcal{T})$ is w.h. and $c(x)\neq c(y)$. Then by Theorem 2.2 $x\in O\in\mathcal{T}$ and $y\in U\in\mathcal{T}$ where O and U are disjoint. Hence $c(x)\subset O$ and $c(y)\subset U$ by Lemma 2.6, and $(X,\mathcal{T})$ is R_1.

Sufficiency. If $(X,\mathcal{T})$ is R_1 and $c(x)=c(y)$, then Theorem 2.2 (a) holds. Otherwise, $c(x)$ and $c(y)$ are contained in disjoint, open sets and Theorem 2.2 (b) holds.

REMARK 2.8. Let $(X,\mathcal{T})$ be any space and define a relation R on X by xRy iff $c(x)=c(y)$. Then $(X/R,\mathcal{T}/R)$ is the well-known T_0-identification with $q: X\to X/R$ the natural projection. (See Willard [7], page 85)

LEMMA 2.9. *If* $(X,\mathcal{T})$ *is* R_0, $q: X\to X/R$ *is an open map.*

PROOF. It suffices to show $O=q^{-1}(q(O))$ for each $O\in\mathcal{T}$. Since $O\subset$

$q^{-1}(q(O))$ is immediate, we let $x \in q^{-1}(q(O))$. Then $q(x)=q(y)$ for some $y \in O$ and thus $x \in c(x) = c(y) \subset O$, verifying the equality.

THEOREM 2.10. *$(X, \mathscr{T})$ is w.h. iff $(X/R, \mathscr{T}/R)$ is Hausdorff.*

PROOF. Necessity. Suppose $(X, \mathscr{T})$ is w.h. and $q(x) \neq q(y)$ in X/R. Then $c(x) \neq c(y)$, and by Theorem 2.2 there are disjoint, open sets O and U such that $x \in O$ and $y \in U$. Thus $q(x) \in q(O) \in \mathscr{T}/R$ and $q(y) \in q(U) \in \mathscr{T}/R$ by Lemmas 2.6 and 2.9, and it remains only to show $q(O) \cap q(U) = \phi$. But if $q(z) \in q(O) \cap q(U)$, then $c(z)=c(x^*)$ and $c(z)=c(y^*)$ for some $x^* \in O$ and $y^* \in U$. Applying Lemma 2.6, $x^* \in O \cap c(z) = O \cap c(y^*) \subset O \cap U$, a contradiction.

Sufficiency. Suppose $S: D \to X$ with $\lim S = x$ and $\lim S = y$. Then $\lim q \circ S = q(x)$ and $\lim q \circ S = q(y)$, implying $q(x)=q(y)$ since $(X/R, \mathscr{T}/R)$ is T_2. Thus $c(x) = c(y)$ and $(X, \mathscr{T})$ is w.h.

3. Some basic properties

THEOREM 3.1. *If $(X, \mathscr{T})$ is w.h. and $Y \subset X$, then $(Y, Y \cap \mathscr{T})$ is w.h.*

PROOF. Let $S: D \to Y$ be a net with $\lim S = y$ and $\lim S = y^*$ in $(Y, Y \cap \mathscr{T})$ and thus in $(X, \mathscr{T})$. Then $c(y)=c(y^*)$ and so $c_Y(y) = Y \cap c(y) = Y \cap c(y^*) = c_Y(y^*)$.

THEOREM 3.2. *If $(X, \mathscr{T})$ is w.h. and $f: (X, \mathscr{T}) \to (Y, \mathscr{U})$ is a homeomorphism, then $(Y, \mathscr{U})$ is w.h.*

PROOF. If $S: D \to Y$ with $\lim S = y$ and $\lim S = y^*$, then $\lim f^{-1} \circ S = f^{-1}(y)$ and $\lim f^{-1} \circ S = f^{-1}(y^*)$. Hence we have $c_X(f^{-1}(y)) = c_X(f^{-1}(y^*))$ and so $c_Y(y) = c_Y(y^*)$.

THEOREM 3.3. Let $(X, \mathscr{T}) = \times \{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$. *Then $(X, \mathscr{T})$ is w.h. iff $(X_\alpha, \mathscr{T}_\alpha)$ is w.h. for all $\alpha \in \Delta$.*

PROOF. Necessity. For each $\alpha \in \Delta$, $(X, \mathscr{T})$ contains a subspace homeomorphic to $(X_\alpha, \mathscr{T}_\alpha)$. Apply Theorems 3.1 and 3.2.

Sufficiency. Let $S: D \to X$ be a net with $\lim S = x$ and $\lim S = y$. Then for each $\alpha \in \Delta$, $\lim P_\alpha \circ S = P_\alpha(x)$ and $\lim P_\alpha \circ S = P_\alpha(y)$ and so $c_\alpha(P_\alpha(x)) = c_\alpha(P_\alpha(y))$. Hence $(X, \mathscr{T})$ is w.h. since $c(x) = \times \{c_\alpha(P_\alpha(x)) : \alpha \in \Delta\} = \times \{c_\alpha(P_\alpha(y) : \alpha \in \Delta\}$ $= c(y)$.

THEOREM 3.4. *A Hausdorff space is weakly Haudorff.*

PROOF. Net limits are unique in a Hausdorff space.

THEOREM 3.5. *A regular space is weakly Hausdorff.*

PROOF. Let $x, y \in X$ and suppose condition (a) of Theorem 2.2 fails. Without loss of generality, we may assume that for some $O \in \mathscr{T}$, $x \in O$ but $y \notin O$. By regularity, there is an $O^* \in \mathscr{T}$ such that $x \in O^* \cup c(O^*) \subset O$, and so $x \in O^*$, $y \in \mathscr{C}c(O^*)$, and $O^* \cap \mathscr{C}c(O^*) = \phi$. This satisfies condition (b), and we conclude the space is w.h.

COROLLARY 3.6. *A pseudometric space is w.h..*

REMARK 3.7. For neither of the two previous theorems does the converse hold. For if $X = \{a, b\}$ and $\mathscr{T} = \{\phi, X\}$, then $(X, \mathscr{T})$ is w.h. but not T_2, while if $(Y, \mathscr{U})$ is a T_2-space which is not T_3 (See Willard [7], Example 14.2), then $(Y, \mathscr{U})$ is w.h. but not regular. We further observe that if we let $(Z, \mathscr{V}) = (X \times Y, \mathscr{T} \times \mathscr{U})$, then $(Z, \mathscr{V})$ is w.h. by Theorem 3.3 but is neither T_2 nor regular. Thus the family of w.h. spaces is strictly larger than the union of all T_2-spaces and all regular spaces.

THEOREM 3.8. *A space is T_2 iff it is T_0 and w.h.*

PROOF. The necessity follows immediately from Theorem 3.4. To prove sufficiency, let $x \neq y$. By the T_0 property, $c(x) \neq c(y)$ and the result follows from condition (b) of Theorem 2.2.

REMARK 3.9. A T_1-space need not be w.h. as seen by an infinite set with the cofinite topology, while a w.h. space need not be T_0 as a two-point, indiscrete space shows.

4. Two structure theorems

DEFINITION 4.1. A space is said to be *saturated* if arbitrary intersections of open sets are open. Equivalently, a space is saturated iff each point has a minimum open neighborhood. (See Lorrain [5])

THEOREM 4.2. *If $(X, \mathscr{T})$ is saturated, the following conditions are equivalent:*
(a) *$(X, \mathscr{T})$ is 0-dimensional*
(b) *$(X, \mathscr{T})$ is completely regular*
(c) *$(X, \mathscr{T})$ is regular*
(d) *$(X, \mathscr{T})$ is w.h.*

(e) $(X, \mathscr{T})$ *is* R_0

(f) $\mathscr{T}=\mathscr{F}$ *(i.e., a set is open iff it is closed.).*

PROOF. That (a) implies (b) implies (c) implies (d) implies (e) follows from Theorem 3.5, Lemma 2.6, and standard results.

(e) implies (f): Suppose $(X, \mathscr{T})$ is R_0 and $O \in \mathscr{T}$, Then for each $x \in O$, $c(x) \subset O$ and so $O = \bigcup \{c(x) : x \in O\}$, a closed set by the saturation property. Thus $\mathscr{T} \subset \mathscr{F}$ and the reverse inclusion follows by complementation.

(f) implies (a): $\mathscr{T}$ itself is an open-closed base for $\mathscr{T}$.

REMARK 4.3. The preceding theorem determines the structure of a saturated, and thus a finite, w.h. topology (i.e., $\mathscr{T}=\mathscr{F}$). To investigate an infinite w.h. topology, we first consider three lemmas.

LEMMA 4.4. *Let $(X, \mathscr{T})$ be a saturated w.h. space, and $x \in O_x \in \mathscr{T}$ where O_x is the minimum open neighborhood of x. Then for any $U \in \mathscr{T}$, either $O_x \cap U = \phi$ or $O_x \subset U$.*

PROOF. *If $O_x \cap U \neq \phi$*, there is a $y \in O_x \cap U$, and we consider Theorem 2.2 with respect to x and y. If condition (b) holds, $x \in O^*$ and $y \in U^*$, where O^* and U^* are disjoint, open sets, and consequently $x \in O_x \cap O^* \in \mathscr{T}$ with $O_x \cap O^* \subsetneqq O_x$, contradicting the minimality of O_x. Thus condition (a) holds, and so $y \in U$ implies $x \in U$, and hence $O_x \subset U$ by minimality.

LEMMA 4.5. *If $(X, \mathscr{T})$ is a saturated w.h. space with O_x the minimum open set containing x and O_y the minimum open set containing y, then either $O_x = O_y$ or $O_x \cap O_y = \phi$.*

PROOF. The result follows by symmetric applications of Lemma 4.4 to O_x and O_y.

LEMMA 4.6. *Let $(X, \mathscr{T})$ be w.h. and $x \in O \in \mathscr{T}$ where O is not the minimum open neighborhood of x. Then there is a $y' \in O$ and there are open sets O' and U' such that* (1) *$x \in O' \subset O$ and $y' \in U' \subset O$ and* (2) *$O' \cap U' = \phi$.*

PROOF. Suppose that for each $y \in O$, condition (a) of Theorem 2.2 holds. Then if $x \in U \in \mathscr{T}$, $y \in U$ for each $y \in O$ and thus $O \subset U$, implying that O is the minimum open neighborhood of x, a contradiction. We conclude that for some $y' \in O$, condition (b) of Theorem 2.2 holds and thus $x \in O^*$ and $y' \in U^*$ for some

O^*, U^* open and disjoint. Then define $O'=O\cap O^*$ and $U'=O\cap U^*$.

THEOREM 4.7. *Let $(X,\mathscr{T})$ be a w.h. space. Then $\mathscr{T}$ is infinite iff there is an infinite family of non-empty, mutually disjoint open sets.*

PROOF. The sufficiency is immediate and the necessity is done by cases.

Case (1) : Suppose $(X,\mathscr{T})$ is saturated. Then for each $x\in X$, there exists O_x, the minimum open neighborhood of x.

We consider the cardinality of $\{O_x : x\in X\}$. Subcase (1) : If $\{O_x : x\in X\}$ is infinite, there is an $X^*\subset X$ such that X^* is infinite and $O_x\neq O_y$ for $x,y\in X^*$ and $x\neq y$. Then by Lemma 4.5, $O_x\cap O_y=\phi$ for $x\neq y$ in X^* and thus $\{O_x : x\in X^*\}$ is the desired family.

Subcase (2) : If $\{O_x : x\in X\}$ is finite, then we can choose representatives $x_1, \cdots, x_n\in X$ such that $\{O_x : x\in X\}=\{O_{x_i} : 1\leq i\leq n\}$ and thus $X=\bigcup\{O_{x_i} : 1\leq i\leq n\}$. Now, letting U be any open set and $J=\{j : U\cap O_{x_j}\neq\phi\}$, we assert that $U=\bigcup\{O_{x_j} : j\in J\}$. For if $x\in U$, then $x\in O_{x_j}$ for some j and thus $j\in J$. Conversely, if $x\in O_{x_j}$ for some $j\in J$, then $O_{x_j}\cap U\neq\phi$ and so $x\in O_{x_j}\subset U$ by Lemma 4.4. It follows that $U=\bigcup(O_{x_j} : j\in J\}$. But since $U\in\mathscr{T}$ was arbitrary and $J\subset\{1,\cdots,n\}$, we conclude $\mathscr{T}$ is finite, a contradiction. So, subcase (2) is impossible and case (1) is proved.

Case (2) : If $(X,\mathscr{T})$ is not saturated, there is an $x\in X$ such that no open neighborhood of x is minimum. We define inductively $\{U_n : n\geq 1\}$ as follows:

Letting $O_0=X$ in Lemma 4.6, there are $y_1\in O_0$ and O_1, $U_1\in\mathscr{T}$ such that $x\in O_1\subset O_0$, $y_1\in U_1\subset O_0$, and $O_1\cap U_1=\phi$. Now $x\in O_1$ and so again by Lemma 4.6, there are $y_2\in O_1$ and $O_2, U_2\in\mathscr{T}$ such that $x\in O_2\subset O_1$, $y_2\in U_2\subset O_1$ and $O_2\cap U_2=\phi$. Continuing in this manner, the family $\{U_n : n\geq 1\}$ is an infinite family of nonempty, mutually disjoint open sets.

5. Compactness

REMARK 5.1. In the following, we shall use the definition of paracompactness given by Gaal [2], who, unlike some authors, does not include T_2 or regular in the definition.

THEOREM 5.2. *Let $(X,\mathscr{T})$ be paracompact. Then $(X,\mathscr{T})$ is w.h. iff $(X,\mathscr{T})$ is regular.*

PROOF. Sufficiency follows from Theorem 3.5. To prove necessity, suppose $(X,\mathscr{T})$ is w.h. and $x\notin F$, a closed set. Then for each $y\in F$, $y\notin\mathscr{C}F$ while

$x \in \mathscr{C} F$ and so by Theorem 2.2 there are disjoint open sets O_y and U_y such that $x \in O_y$ and $y \in U_y$. Now $X = \mathscr{C} F \cup \bigcup \{U_y : y \in F\}$ and so there is an open, locally-finite refinement $\{V_\beta : \beta \in \Gamma\}$ which covers X. Letting $V = \bigcup \{V_\beta : V_\beta \cap F \neq \phi\}$, we have $F \subset V$ and we assert $x \in \mathscr{C}\, c(V)$. For otherwise, $x \in c(V) = \bigcup \{c(V_\beta) : V_\beta \cap F \neq \phi\}$ by local finiteness, and so $x \in c(V_\beta{}^*)$ where $V_\beta{}^* \cap F \neq \phi$. Thus $V_\beta{}^* \not\subset \mathscr{C} F$ and so $V_\beta{}^* \subset U_y{}^*$ for some $y^* \in F$. But then $x \in c(V_\beta{}^*) \subset c(U_y{}^*)$, a contradiction since $x \in O_y{}^*$ and $O_y{}^* \cap U_y{}^* = \phi$. We conclude that $F \subset V \in \mathscr{T}$ and $x \in \mathscr{C}\, c(V) \in \mathscr{T}$ with $V \cap \mathscr{C}\, c(V) = \phi$. Hence $(X, \mathscr{T})$ is regular.

COROLLARY 5.3. *A compact space is w.h. iff it is regular.*

PROOF. A compact space is paracompact.

COROLLARY 5.4. *A paracompact, w.h. space is normal.*

PROOF. A paracompact, w.h. space is paracompact and regular, and thus normal.

COROLLARY 5.5. *A compact, w.h. space is completely regular and normal.*

PROOF. Regularity follows from Corollary 5.3 and normality from Corollary 5.4. Thus the space is completely regular as well.

THEOREM 5.6. *In a w.h. space, closures of compact sets are compact.*

PROOF. Let $(X, \mathscr{T})$ be w.h. and $K \subset X$ be compact. Then if $c(K) \subset \bigcup \{O_\alpha : \alpha \in \Delta\}$, an open cover, we have $K \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Now if $x \in c(K)$, there is a net $S : D \to K$ such that $\lim S = x$. Hence, by compactness, there is a subnet $T : D \to K$ such that $\lim T = y$ for some $y \in K$. Since $\lim T = x$ and $\lim T = y$, $c(x) = c(y)$ and thus, by Lemma 2.6, $x \in c(x) = c(y) \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. It follows that $c(K) \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$.

6. Local compactness

REMARK 6.1. There are at least three definitions of local compactness appearing in the literature. Kelley [3] requires that each point have a compact neighborhood, while Willard [7] demands that the topology have a base consisting of compact neighborhoods. Royden [6] requires each point to be contained in an open set whose closure is compact, although some authors refer to this variant as strong local compactness. We shall denote these properties by l.c., c.n.b. and s.l.c., respectively. Examples can be constructed to show these are all different, although a truly confusing situation is averted since the concepts agree in Hausdorff or regular spaces. In fact, they agree in w.h. spaces.

THEOREM 6.2. *A w.h. space $(X, \mathscr{T})$ is l.c. iff it is s.l.c..*

PROOF. Since the sufficiency always holds, we need only prove necessity. So for each $x \in X$, $x \in O \subset N$ where $O \in \mathscr{T}$ and N is comact. Then $x \in O \subset c(O) \subset c(N)$. But $c(N)$ is compact by Theorem 5.6 and so $c(O)$ is compact also.

THEOREM 6.3. *$(X, \mathscr{T})$ is w.h. and l.c. iff $(X^*, \mathscr{T}^*)$ is w.h. where $(X^*, \mathscr{T}^*)$ is the one-point compactification of $(X, \mathscr{T})$.*

PROOF. Necessity. We shall use Theorem 2.2.

Case(1) : Let $x, y \in X$ and suppose condition (b) of Theorm 2.2 fails in $(X^*, \mathscr{T}^*)$. Then it fails in $(X, \mathscr{T})$ as well and so $x \in O \in \mathscr{T}$ iff $y \in O \in \mathscr{T}$. Hence $x \in O^* \in \mathscr{T}^*$ iff $y \in O^* \in \mathscr{T}^*$.

Case (2) : Let $x \in X$. By Theorem 6.2 $x \in O \subset c(0)$ where $O \in \mathscr{T}$ and $c(O)$ is X-compact. Hence $\infty \in \{\infty\} \cup \mathscr{C}_X c(O) \in \mathscr{T}^*$ and $x \in O \in \mathscr{T}^*$ with $O \cap (\{\infty\} \cup \mathscr{C}_X\, c(O)) = \phi$.

Sufficiency. If $(X^*, \mathscr{T}^*)$ is w.h., $(X, \mathscr{T})$ is w.h. by Theorem 3.1. Moreover, for $x \in X \in \mathscr{T}^*$, Theorem 2.2 implies $x \in O^* \in \mathscr{T}^*$ and $\infty \in U^* \in \mathscr{T}^*$ for some O^*, U^* disjoint. But then $U^* = \{\infty\} \cup U$ where $U \in \mathscr{T}$ and $\mathscr{C}_X U$ is X-compact, and so $x \in O^* \subset \mathscr{C}_X U$ implies $(X, \mathscr{T})$ is l.c.

COROLLARY 6.4. *An l.c., w.h. space $(X, \mathscr{T})$ is completely regular.*

PROOF. By the previous theorem $(X^*, \mathscr{T}^*)$ is w.h. and thus completely regular by Corollary 5.5. Hence $(X, \mathscr{T})$ is also completely regular.

THEOREM 6.5. *The three types of local compactness are equivalent in w.h. spaces.*

PROOF. In light of Theorem 6.2 and the fact that c.n.b. always implies l.c., it suffices to show that if $(X, \mathscr{T})$ is w.h. and l.c., it is c.n.b. So let $x \in O \in \mathscr{T}$. By Corollary 6.4 $(X, \mathscr{T})$ is regular and thus $x \in O^* \subset c(O^*) \subset O$ for some $O^* \in \mathscr{T}$. Moreover, by Thorem 6.2 there is a $U \in \mathscr{T}$ such that $x \in U$ and $c(U)$ is compact. Then $x \in O^* \cap U \subset c(0^*) \cap c(U) \subset O$ and $c(O^*) \cap c(U)$ is the required compact neighborhood.

7. Complete regularity

REMARK 7.1. It is routine to show that if $f: (X, \mathscr{T}) \to [0, 1]$ is continuous and we define a pseudometric on X by $d(x, y) = |f(x) - f(y)|$, then the identity $I: (X, \mathscr{T}) \to (X, \mathscr{T}(d))$ is continuous.

LEMMA 7.2. *If d is as above, (X,d) is totally bounded.*

PROOF. $f(X)\subset[0,1]$ is totally bounded and so for $\varepsilon>0$, there are $x_1,\cdots,x_n\in X$ such that for each $x\in X$, $d(x,x_i)=|f(x)-f(x_i)|<\varepsilon$ for some i.

REMARK 7.3. Let (X,d) be as above and (X^*,d^*) be the usual pseudometric completion. That is, X^* is the set of all d-Cauchy sequences from X and $d^*((x_n),(y_n))=\lim_n d(x_n,y_n)$. Let $F:X\to X^*$ be the natural isometry $F(x)=(x,x,x,\cdots)$.

LEMMA 7.4. *Using the terminology above, $(X^*,\mathscr{T}(d^*))$ is compact.*

PROOF. The complete space (X^*,d^*) is totally bounded since $F(X)$ is dense and totally bounded by Lemma 7.2.

REMARK 7.5. Let $(X,\mathscr{T})$ be completely regular and let $\mathscr{S}=\{f_\alpha:\alpha\in\Delta\}$ be the indexed family of all continuous maps from X to $[0,1]$. For each $\alpha\in\Delta$, define a pseudometric d_α on X as in Remark 7.1. Then $I_\alpha:(X,\mathscr{T})\to(X_\alpha,\mathscr{T}(d_\alpha))$ is continuous where $X_\alpha=X$ for all α and I_α is the identity. Also, let (X_α^*,d_α^*) be the completion of (X_α,d_α) and $F_\alpha:X_\alpha\to X_\alpha^*$ the natural isometry as in Remark 7.3.

LEMMA 7.6. *Using the notation of Remark 7.5, $\mathscr{F}=\{F_\alpha\circ I_\alpha:\alpha\in\Delta\}$ is a family of continuous functions which separate points and separate points and closed sets.*

PROOF. For each α, $F_\alpha\circ I_\alpha:X\to X_\alpha^*$ is continuous since I_α is continuous and F_α is an isometry. Secondly, if $x\neq y$ and $\alpha\in\Delta$ is arbitrary, $F_\alpha\circ I_\alpha(x)\neq F_\alpha\circ I_\alpha(y)$. Thus $\mathscr{F}$ separates points. Finally, if $x\notin F$ where F is closed in $(X,\mathscr{T})$, there is an $f_\beta\in\mathscr{S}$ such that $f_\beta(x)=0$ and $F_\beta(F)\subset\{1\}$. We assert that $F_\beta\circ I_\beta(x)\notin c_\beta(F_\beta\circ F_\beta(F))$ and it suffices to show $\{x^*\in X_\beta^*:d_\beta^*(x^*,F_\beta\circ I_\beta(x))<1/2\}\cap F_\beta\circ I_\beta(F)=\phi$. But if we deny this statement, then for some $y\in F$, $1/2>d_\beta^*(F_\beta\circ I_\beta(y),\ F_\beta\circ I_\beta(x))=d_\beta(x,y)=|f_\beta(x)-f_\beta(y)|$, a contradiction. Thus $\mathscr{F}$ separates points and closed sets.

THEOREM 7.7. *$(X,\mathscr{T})$ is completely regular iff $(X,\mathscr{T})$ is homeomorphic to a subspace of a compact, w.h. space.*

PROOF. Necessity. Letting $\mathscr{F}$ be as above, the Embedding Lemma (Kelley [3], theorem 4.5) implies that $(X,\mathscr{T})$ is homeomorphic to a subspace of $\times\{X_\alpha^*,\mathscr{T}(d_\alpha^*)):\alpha\in\Delta\}$, this product being compact by Lemma 7.4 and the

Tychonoff theorem, and w.h. by Theorem 3.3 and Corollary 3.6.

The sufficiency follows immediately from Corollary 5.5.

COROLLARY 7.8. *A space is completely regular iff it is homeomorphic to a subspace of a compact, regular space.*

PROOF. Apply Corollary 5.3 to the preceding theorem.

COROLLARY 7.9. *A space is completly regular iff it has a w.h. compactification.*

PROOF. The necessity follows from Theorems 3.1 and 7.7 while the sufficiency follows from Corollary 5.5.

8. Some conclusions

We have seen that the concept of weakly Hausdorff, although weaker than T_2 or regular, preserves many of the properties of such spaces, and important results become Corollaries (for instance, replace w.h. by T_2 or regular in 4.7, 5.4, 5.5, 5.6, 6.4, and 6.5). Moreover, in light of Theorem 5.2 and Corollary 5.3, the w.h. property provides a characterization of regularity on compact or paracompact spaces solely in terms of nets, or, by Theorem 2.2, in terms of points. Finally, w.h. spaces are seen to arise naturally in topology in Theorems 2.10 and 7.7 and in Corollary 7.9.

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REFERENCES

[1] A.S. Davis, *Indexed Systems of Neighborhoods for General Topological Spaces*, The American Mathematical Monthly, Vol. 68 (1961) pp. 886~893.

[2] S. Gaal, *Point Set Topology*, Academic Press, 1964.

[3] J. Kelley, *General Topology*, Van Nostrand, 1955.

[4] N. Levine, *Spaces in Which the Closure of a Compact Set is Compact*, to appear.

[5] F. Lorrain, *Notes on Topological Spaces with Minimum Neighborhoods*, The American Mathematical Monthly, Vol. 76 (1969) pp. 616~627.

[6] H.L. Royden, *Real Analysis*, The Macmillan Company, 1968.

[7] S. Willard, *General Topology*, Addison-Wesley Publishing Company, 1968.