Kyungpook Math. J.
Volume 15, Number 1
June, 1975

ON PAIRWISE s-NORMAL SPACES

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1. Introduction

A subset A of a topological space X is semi open [3] if for some open set O, $O\subset A\subset \mathrm{cl}\, O$, where $\mathrm{cl}\, O$ denotes the closure of O in X. Every open set is semi open but the converse may be false [3]. The complement of a semi open set is called semi closed. A point $p\in X$ is a semi limit point of $A\subset X$, if every semi open set containing p contains a point of A distinct from p. The union of A and the set of all the semi limit points of A is called the semi closure [1] of A. We denote it by scl A. Infact, it is the smallest semi closed set containing A, and A is semi closed iff $A=\mathrm{scl}\, A$. A bitopogical space (X,P_1,P_2) is pairwise normal [2] if for every P_i-closed set A and P_j-closed set B such that $A\cap B=\phi$ there exist a P_j-open set U and a P_i-open set V such that $A\subset U$, $B\subset V$ and $U\cap V=\phi : i,j=1,2,\ i\neq j$. Also a bitopological space (X,P_1,P_2) is pairwise R_0 [6], if for every P_i-open set G, $x\in G$ implies that $P_j\text{-cl}\{x\}\subset G : i\neq j,\ i,j=1,2$. Lastly, a bitopological space (X,P_1,P_2) is pairwise s-regular [4] if for every P_i-closed set F and a point $x\notin F$ there exists a P_j-semi open set U and a P_i-semi open set V such that $U\cap V=\phi$, $F\subset U$, $x\in V : i\neq j,\ i,j=1,2$. Every pairwise regular space [2] is pairwise s-regular but the converse may not be true [4].

In the present paper we introduce the concept of pairwise s-normality which is strictly weaker than pairwise normality. At the same time it also presents the role of semi open sets in topology. Throughout the paper $X\sim B$ denotes the complement of B in X.

2. Pairwise s-normal spaces

DEFINITION. A bitopological space(X,P_1,P_2) is ***pairwise s-normal*** if for every P_i-closed set A and P_j-closed set B such that $A\cap B=\phi$, there exist a P_j-semi open set U and P_i-semi open set V such that $A\subset U$, $B\subset V$ and $U\cap V=\phi : i,j=1,2$ $i\neq j$.

It is evident that every pairwise normal space is pairwise s-normal. However, the converse may be false.

EXAMPLE 1. Let $X=\{a,b,c,d\}$,

$P_1=\{\phi, \{b\}, \{d\}, \{b,d\}, \{a,b,d\}, \{b,c,d\}, X\}$,

$P_2=\{\phi, \{b\}, \{d\}, \{b,d\}, \{b,c,d\}, X\}$.

Then, (X,P_1,P_2) is pairwise s-normal but it is not pairwise normal. Also, note that it is not pairwise R_0.

REMARK 1. Pairwise s-normality need not imply pairwise s-regularity.

EXAMPLE 2. Let $X=\{a,b,c\}$,

$P_1=\{\phi, \{a\}, \{b,c\}, X\}$,

$P_2=\{\phi, \{b\}, \{b,c\}, X\}$.

Then, (X,P_1,P_2) is pairwise s-normal (infact, pairwise normal) but it is not pairwise s-regular.

EXAMPLE 3. Let X be the set of all the real numbers, P_1 be the countable complement topology and P_2 be the usual topology. Then the space (X,P_1,P_2) is pairwise R_0. But it is neither pairwise s-regular nor pairwise s-normal. For, every P_2-open set is uncountable and so intersects every nonempy P_1-open set.

Therefore every P_1-semi open set meets every P_2-semi open set.

REMARK 2. The axioms of pairwise R_0 and pairwise s-normal are independent (Example 1 and 3). A pairwise R_0-space may fail to be pairwise s-regular (Example 3).

However, we have the following theorem.

THEOREM 1. *Every pairwise s-normal pairwise R_0-space (X,P_1,P_2) is pairwise* s-regular.

PROOF. Let F be a P_i-closed set $i=1$ or 2 and let $x\in X$ be such that $x\notin F$. Now X being pairwise R_0, $P_j\text{-cl}\{x\}\cap F=\phi$, $j=1,2$, $i\neq j$. The theorem now follows by pairwise s-normality of X.

THEOREM 2. *In a bitopological space (X,P_1,P_2) the following conditions are equivalent*:

(a) *(X,P_1,P_2) is pairwise s-normal.*

(b) *For each P_i-closed set A and P_j-open set B containing A there is a P_j-semi open set U such that $A\subset U\subset P_i\text{-scl } U\subset B$, $i\neq j$, $i,j=1,2$.*

(c) *For each P_i=closed set A and P_j-closed set B disjoint from A there exists a*

P_j-semi open set U containing A and such that $(P_i\text{-scl } U)\cap B=\phi$, $i\neq j$, $i,j=1,2$.

KROOF. (a) $\Rightarrow$ (b) : Since A is P_i-closed and disjoint from a P_j-closed set $X\sim B$, there exist disjoint semi open sets U and V such that U is P_j-semi open and V is P_i-semi open such that $A\subset U$, $X\sim B\subset V$. And so, $A\subset U\subset X\sim V\subset B$. Hence, $A\subset U\subset P_i\text{-scl } U\subset B$.

(b) $\Rightarrow$ (c). Since $X\sim B$ is P_j-open and contains a P_i-closed set $A, i\neq j, i,j=1,2$, there exists a P_j-semi open set U such that $A\subset U\subset P_i\text{-scl } U\subset X\sim B$. Clearly, $(P_i\text{-scl } U)\cap B=\phi$.

(c) $\Rightarrow$ (a). Let A is P_i-closed, B is P_j-closed and $A\cap B=\phi$. Then there exists a P_j-semi open set U such that $A\subset U$ and $(P_i\text{-scl } U)\cap B=\phi$. Then $X\sim(P_i\text{-scl } U)$ is P_i-semi open set containing B and disjoint from U.

REMARK 3. Pairwise s-normality is not hereditary. Even a biclosed bisemiopen subspace of a pairwise s-normal space may not be pairwise s-normal. For, $\{a, b, c\}$ is a biclosed bisemiopen subspace of the pairwise s-normal space X of example 1 but it is not pairwise s-normal. However,

THEOREM 3. *Every biopen and biclosed subspace of a pairwise s-normal space (X, P_1, P_2) is pairwise s-normal.*

In the proof of this theorem we use the the following result.

LEMMA. [5] *If Y is open and A is semi open in a topological space X then $Y\cap A$ is semi open in Y.*

PROOF of the theorem. Let (Y, T_1, T_2) be a biclosed biopen subspace of (X, P_1, P_2). Let A and B be any two disjoint sets in Y such that A is T_i-closed and B is T_j-closed, $i\neq j$, $i,j=1,2$. Since Y is biclosed in X, A is P_i-closed and B is P_j-closed. Now, X being pairwise s-normal there exist P_j-semi open set U and P_i-semi open set V such that $A\subset U$, $B\subset V$ and $U\cap V=\phi$. Since Y is biopen in X it follows by the lemma that $U\cap Y$ is T_j-semi open, $V\cap Y$ is T_i-semi open. It is clear that $U\cap Y$ and $V\cap Y$ are disjoint and contain A and B respectively. Consequently, (Y, T_1, T_2) is pairwise s-normal by definition.

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