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# PRODUCTS OF PAIRWISE COMPACT SPACES

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## 1. Introduction

Several definitions of pairwise compactness for the bitopological spaces of Kelly [2] have appeared in the literature. Of these definitions, this paper will concern those of Kim [3] ($K$-pairwise compactness) and Fletcher, Hoyle and Patty [1] ($FHP$-pairwise compactness) showing their equivalence for pairwise Hausdorff spaces.

Pahk and Choi [4] gave a characterization of $FHP$-pairwise compact spaces in terms of convergence of pairwise filter bases. A modified form of a theorem of Pahk and Choi will be used to establish necessary and sufficient conditions for $FHP$-pairwise compactness of products of bitopological spaces. The reader is referred to the above sources for the appropriate definitions.

## 2. Products of bitopological spaces

In a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ a filterbase on $X$ is termed a pairwise filterbase by Pahk and Choi if $\mathscr{F}$ contains a proper subset of $X$ which is not dense in $X$ with respect to either $\mathscr{P}$ or $\mathscr{Q}$. If it is noted that any filterbase on $X$ that is not a pairwise filterbase must accumulate at every point with respect to one of the topologies, theorem 2.1 of Pahk and Choi can be restated in a slightly different form.

THEOREM 2.1. *The following statements about a bitopological space are equivalent.*

(a) $(X, \mathscr{P}, \mathscr{Q})$ *is FHP-pairwise compact*

(b) *Each pairwise closed family of subsets of $X$ satisfying the finite intersection property has nonvoid intersection.*

(c) *For each filterbase on $X$ there exists at least one point which is both a $\mathscr{P}$-accumulation and a $\mathscr{Q}$-accumulation point or the filterbase accumulates at no point with respect to one topology and at every point with respect to the other topology.*

(d) *For each maximal filterbase on $X$ there exists at least one point which is both a $\mathscr{P}$-limit point and a $\mathscr{Q}$-limit point or the filterbase converges to no point with respect to one topology and to every point with respect to the other topology.*

THEOREM 2.2. *If a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is K-pairwise compact, each ultrafilter has a limit point with respect to each topology or converges to no point with respect to one topology and to every point with respect to other topology.*

PROOF. Assume some ultrafilter $\mathscr{F}$ on $X$ has no $\mathscr{P}$-limit point. If there exists some $y$ in $X$ which is not a $\mathscr{Q}$-limit point of $\mathscr{F}$, then some $\mathscr{Q}$-open neighborhood $V$ of $y$ is not in $\mathscr{F}$. Consider the adjoint topology $\mathscr{P}(V)=\{\phi, X\} \cup \{U \cup V \mid U \in \mathscr{P}\}$. Each point $x$ of $X$ has a $\mathscr{P}$-open neighborhood $U_x$ such that $U_x$ is not in $\mathscr{F}$. But then $\{U_x \cup V\}$ is a cover of $X$ by members of $\mathscr{P}(V)$ having no finite subcover and $\mathscr{P}(V)$ can not be compact, hence $(X, \mathscr{P}, \mathscr{Q})$ is not $K$-pairwise compact.

COROLLARY 2.1. *K-pairwise compactness is equivalent to FHP-pairwise compactness in the category of pairwise Hausdorff bitopological spaces.*

PROOF. Swart [5] proved that $FHP$-pairwise compactness implies $K$-pairwise compactness. The reverse implication follows from theorems 2.1 and 2.2 and a result of Pahk and Choi that a bitopological space is pairwise Hausdorff if and only if no filter has distinct $\mathscr{P}$ and $\mathscr{Q}$-limit points.

An immediate result of theorems 2.1 and 2.2 is that for noncompact topologies, pairwise compactness implies very strong connectivity properties.

COROLLARY 2.1. *If $(X, \mathscr{P}, \mathscr{Q})$ is K-(FHP-) pairwise compact and $(X, \mathscr{P})$ is not compact, then $\mathscr{Q}$ is hyperconnected.*

PROOF. If $(X, \mathscr{P})$ is not compact, then some ultrafilter $F$ is not $\mathscr{P}$-convergent so must $\mathscr{Q}$ converge to each point and $\mathscr{Q} \subset \mathscr{F}$ is hyperconnected.

THEOREM 2.3. *If $(X, \mathscr{P})$ is a compact Hausdorff space and $\mathscr{Q}$ is any topology on $X$ for which $(X, \mathscr{P}, \mathscr{Q})$ is pairwise Hausdorff and FHP-pairwise compact, then $\mathscr{P}=\mathscr{Q}$.*

PROOF. If $\mathscr{F}$ is an ultrafilter which $\mathscr{P}$-converges to $x$, $(X, \mathscr{P})$ Hausdorff and $FHP$-pairwise compactness of $(X, \mathscr{P}, \mathscr{Q})$ implies $\mathscr{F}$ must $\mathscr{Q}$-converge to $x$. Conversely, if $\mathscr{F}$ $\mathscr{Q}$-converges to $x$, compactness of $(X, \mathscr{P})$ implies $\mathscr{P}$-convergence of $\mathscr{F}$ and $(X, \mathscr{P}, \mathscr{Q})$ pairwise Hausdorff implies $\mathscr{F}$ $\mathscr{P}$-converges to $x$.

If $\{(X_i, \mathscr{P}_i, \mathscr{Q}_i)\}$ is a family of bitopological spaces, let $\mathscr{P}$ and $\mathscr{Q}$ represent the product topologies on $\prod X_i$ determined by $\{\mathscr{P}_i\}$ and $\{\mathscr{Q}_i\}$. Swart [5] has shown that $(\prod X_i, \mathscr{P}, \mathscr{Q})$ being $FHP$-pairwise compact implies $FHP$-pairwise compactness of each bitopological space $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$. A somewhat more general result is presented here.

THEOREM 2.4. *If* $\{(X_i, \mathscr{P}_i, \mathscr{Q}_i) | i \in I\}$ *is a family of bitopological spaces with* $|I|>1$, $(\Pi X_i, \mathscr{P}, \mathscr{Q})$ *is FHP-pairwise compact if and only if each* $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ *is FHP-pairwise compact and if some* $(X_j, \mathscr{P}_j)$ $((X_j, \mathscr{Q}_j))$ *is not compact* $(X_i, \mathscr{Q}_i)$ $((X_i, \mathscr{P}_i))$ *is indiscrete for each* $i \neq j$.

PROOF. As noted above, *FHP*-pairwise compactness of $(\Pi X_i, \mathscr{P}, \mathscr{Q})$ implying *FHP*-pairwise compactness of each $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ was established by Swart [5].

Suppose some $(X_j, \mathscr{P}_j)$ is not compact. Then some ultrafilter $\mathscr{F}_j$ on $X_j$ has no $\mathscr{P}_j$-limit points. Then if $k \neq j$ and $\mathscr{F}_k$ is any ultrafilter on $X_k$, consider an ultrafilter $\mathscr{G}$ on $\Pi X_i$ containing the filterbase $\Pi \mathscr{G}_i = \{\Pi A_i | A_i \in \mathscr{G}_i\}$ where $\mathscr{G}_j = \mathscr{F}_j$, $\mathscr{G}_k = \mathscr{F}_k$ and $\mathscr{G}_i$ is any ultrafilter on $X_i$ for $i \neq j$, $k$. Given any projection map $\mathscr{P}_i$, $\mathscr{P}_i(\mathscr{G}) = \mathscr{G}_i$. Since $\mathscr{P}_j(\mathscr{G}) = \mathscr{F}_j$ does not $\mathscr{P}_j$-converge, $\mathscr{G}$ does not $\mathscr{P}$-converge and must $\mathscr{Q}$-converge to every point of $\Pi X_i$. Therefore, $\mathscr{F}_k$ must $\mathscr{Q}_k$-converge to each point of $X_k$. since $\mathscr{F}_k$ was an arbitrary ultrafilter on $X_k$, $X_k$ must be indiscrete.

If the conditions hold, assume $\mathscr{F}$ is an ultrafilter on $\Pi X_i$. If $\mathscr{F}$ has no $\mathscr{P}$-limit points, then some $\mathscr{P}_i(\mathscr{F})$ does not $\mathscr{P}_i$-converge and $(X, \mathscr{P}_i)$ is not compact. Then *FHP*-pairwise compactness of $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ and $(X_j, \mathscr{Q}_j)$ indiscrete for $i \neq j$ implies $\mathscr{F}$ $\mathscr{Q}$-converges to each point of $\Pi X_i$. If $\mathscr{F}$ both $\mathscr{P}$ and $\mathscr{Q}$-converges, since each $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ is *FHP*-pairwise compact, each $P_i(\mathscr{F})$ has a common $\mathscr{P}_i$ and $\mathscr{Q}_i$ limit point and $\mathscr{F}$ has a common $\mathscr{P}$ and $\mathscr{Q}$ limit point so $(\Pi X_i, \mathscr{P}, \mathscr{Q})$ is *FHP*-pairwise compact.

COROLLARY 2.3. *In the category of pairwise Hausdorff spaces, if* $|I|>1$, $(\Pi X_i, \mathscr{P}, \mathscr{Q})$ *is FHP-pairwise compact if and only if each* $\mathscr{P}_i = \mathscr{Q}_i$ *and* $\mathscr{P}_i$ *is compact.*

PROOF. If the product space is *FHP*-pairwise compact and some $\mathscr{P}_i$ is not compact, then each $\mathscr{Q}_j$ is indiscrete for $j \neq i$ and $(X_j, \mathscr{P}_j, \mathscr{Q}_j)$ cannot be pairwise Hausdorff. Since each $(X_i, \mathscr{P}_i)$ and $(X_i, \mathscr{Q}_i)$ are compact and $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ is pairwise Hausdorff, $\mathscr{P}_i = \mathscr{Q}_i$ by theorem 10 of Fletcher, Hoyle and Patty [1].

It may be easily observed that for any topological space $(X, \mathscr{P})$, if $\mathscr{Q}$ is the finite complement topology on $X$, then $(X, \mathscr{P}, \mathscr{Q})$ is *FHP*-pairwise compact.

Therefore, for any topology $\mathscr{P}$ on a set $X$ there exists at least $T_1$ topology $\mathscr{Q}$ for which$(X, \mathscr{P}, \mathscr{Q})$ is $FHP$-pairwise compact. Since there exist maximal $T_1$-compact topologies on any set, a corresponding question arises for bitopological spaces: Given a topological space $(X, \mathscr{P})$, what conditions on $\mathscr{P}$ are necessary and sufficient to assure the existence of a maximal topology $\mathscr{Q}$ on $X$ such that $(X, \mathscr{P}, \mathscr{Q})$ is $FHP$-pairwise compact? Such a topology would of necessity be $T_1$.

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