Kyungpook Math. J.
Volume 15, Number 1
June, 1975

A NOTE ON PROXIMITIES AND PAIRWISE REGULAR SPACES

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1. Introductory concepts

Let X be a non-empty set. For a binary relation δ defined on $P(X)$, the power set of X, consider the following axioms:

1. $(A, B) \in \delta \Longleftrightarrow (B, A) \in \delta$,
2. $(A \cup B, C) \in \delta \Longleftrightarrow (A, C) \in \delta$ or $(B, C) \in \delta$,

2′ $(A, B \cup C) \in \delta \Longleftrightarrow (A, B) \in \delta$ or $(A, C) \in \delta$,

3. $A \cap B \neq \phi \Longrightarrow (A, B) \in \delta$,
4. $(A, B) \in \delta \Longrightarrow A \neq \phi \neq B$,
5. $(A, B) \notin \delta \Longrightarrow \exists E \in P(X)$ such that $(A, E) \notin \delta$ and $(X-E, B) \notin \delta$,

5′. $(x, B) \notin \delta \Longrightarrow \exists E \in P(X)$ such that $(x, E) \notin \delta$ and $(X-E, B) \notin \delta$,

5″. $(A, B) \in \delta$ and $(b, C) \in \delta$ for all $b \in B \Longrightarrow (A, C) \in \delta$.

A relation δ satisfying 1, 2, 3, 4 and 5 above is said to be an *EF-proximity*. δ satisfying 2, 2′, 3, 4 and 5 is called a *quasi-proximity*. If δ satisfies 1, 2, 3, 4 and 5′ then we call it a *local proximity* where as δ satisfying 2, 2′, 3, 4 and 5′ is termed as a *PR-proximity*. Similarly if δ satisfies the set of axioms 1, 2, 3, 4 and 5″ (respectively 2, 2′, 3, 4 and 5″) then it is called an *LO*-(respectively an *LE*-) *proximity*. If δ satisfies the axiom

6. $(x, y) \in \delta \Longrightarrow x = y$,

then it is said to be *separated*.

If δ is any proximity on X, then δ^{-1} defined by

$$(A, B) \in \delta^{-1} \text{iff } (B, A) \in \delta,$$

is also a proximity of the same type. We call δ^{-1} the *conjugate* of δ.

To each δ, there is associated a topology $\mathscr{T}(\delta) = \{A \subset X \mid x \in A \Longrightarrow (x, X-A) \notin \delta\}$. We call $\mathscr{T}(\delta)$, the topology *induced* by δ. It is known that if δ is an *EF*-(respectively Local, *LO*-) proximity then $\mathscr{T}(\delta)$ is completely regular (respectively regular, R_0) and conversely given any completely regular (respectively regular, R_0) topology $\mathscr{T}$ on X it is possible to define an *EF*-(respectively local, *LO*-) proximity on X such that $\mathscr{T}(\delta) = \mathscr{T}$. If δ is a quasi (respectively, *PR*-, *LE*-) proximity on X, then $\mathscr{T}(\delta)$ need not satisfy any separation axiom. Further given any topology $\mathscr{T}$ on X it is possible to define a quasi-(PR-, LE-) proximity on X such that $\mathscr{T}(\delta)=\mathscr{T}$, $\mathscr{T}(\delta)$ is T_1 iff δ is separated.

Given two proximities δ_1 and δ_2 on X, we say δ_1 is *finer* than δ_2 and write $\delta_1 \geq \delta_2$ if $(A, B) \in \delta_1 \Longrightarrow (A, B) \in \delta_2$. It is known that finer proximities induce finer topologies.

If $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on X, then the ordered triple $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called a bitopological space [1]. It is said to be (a) *pairwise* R_0 [5] if for every $x \in G \in \mathscr{T}_1$, $\mathscr{T}_2$-cl$\{x\} \subset G$ and $x \in G \in \mathscr{T}_2 \Longrightarrow \mathscr{T}_1$-cl$\{x\} \subset G$. (b) *pairwise regular* [1] if every point of X has $\mathscr{T}_1$ neighbourhood base consisting of $\mathscr{T}_2$-closed sets and a $\mathscr{T}_2$ neighbourhood base consisting of $\mathscr{T}_1$-closed sets; (c) *pairwise completely regular* [3] if for each $\mathscr{T}_i$ closed set F and $x \in X-F$ there exists $f: X \longrightarrow [0,1]$ which is, $\mathscr{T}_i$-upper semi-continuous (usc) and $\mathscr{T}_j$-lower semi-continuous (lsc) such that $f(x)=0$ and $f(F)=1$ $(i \neq j, i, j=1,2)$: (d) *pairwise normal* [1] if for every $\mathscr{T}_1$-closed set A and $\mathscr{T}_2$-closed set B with $A \cap B=\phi$ there exist a $\mathscr{T}_2$-open set U containing A and $\mathscr{T}_1$-open set V containing B such that $U \cap V=\phi$.

Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space and f a real valued function on X which is $\mathscr{T}_1$-usc and $\mathscr{T}_2$-lsc, then $\{x \in X : f(x) \leq 0\}$ is called a $\mathscr{T}_2$*-zero set* (w.r.t. $\mathscr{T}_1$) and $\{x \in X : f(x) \geq 0\}$ is called a $\mathscr{T}_1$*-zero set* (w.r.t. $\mathscr{T}_2$). The set $\{x \in X : f(x) > 0\}$ is called a $\mathscr{T}_2$*-cozero set* (w.r.t. $\mathscr{T}_1$) and $\{x \in X : f(x) < 0\}$ is called a $\mathscr{T}_1$*-cozero set* (w.r.t. $\mathscr{T}_2$). Similar definitions are made in case when f is $\mathscr{T}_1$-lsc and $\mathscr{T}_2$-usc. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise completely regular iff $\mathscr{T}_1$-zero sets form a base for $\mathscr{T}_1$-closed sets and $\mathscr{T}_2$-zero sets form a base for $\mathscr{T}_2$-closed sets.

2. Pairwise R_0 and pairwise regular spaces

If δ is an LE-(Quasi-or PR-) proximity on X, then $(X, \mathscr{T}(\delta), \mathscr{T}(\delta^{-1}))$ is a bitopological space. Lane [4] has shown that this bitopological space is pairwise completely regular if δ is a quasi-proximity. He also proved that given a pairwise completely regular space $(X, \mathscr{P}, \mathscr{Q})$, there exists a quasi-proximity δ on X such that $\mathscr{T}(\delta)=\mathscr{P}$ and $\mathscr{T}(\delta^{-1})=\mathscr{Q}$. A proximity satisfying the later condition is said to be a *compatible* proximity on $(X, \mathscr{P}, \mathscr{Q})$. Here we consider similar problem when δ is an LE and a PR- proximity.

THEOREM 2.1. *If δ is an LE-proximity, when $(X, \mathscr{T}(\delta), \mathscr{T}(\delta^{-1}))$ is necessarily pairwise R_0. Conversely a compatible LE-proximity can be defined on a pairwise R_0-space $(X, \mathscr{P}, \mathscr{Q})$ by setting*

$$(A, B) \in \delta \textit{ iff } \mathscr{Q}\text{-cl } A \cap \mathscr{P}\text{-cl } B \neq \phi.$$

PROOF. That $(X, \mathscr{T}(\delta), \mathscr{T}(\delta^{-1}))$ is pairwise R_0 follows from the fact that $(x, y) \in \delta \Longleftrightarrow (y, x) \in \delta^{-1}$(Theorem 3.3 of [2]). To see that δ defined above is an *LE*-proximity it is sufficient to verify the axiom 5″ only. Let $(A, B) \in \delta$ and $(b, C) \in \delta$ for each $b \in B$. Then $\mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $B \neq \phi$ and $\mathscr{Q}$-cl $\{b\} \cap P$-cl $C \neq \phi$ for each $b \in B$, i.e., there exists a $c \in \mathscr{P}$-cl C such that $c \in \mathscr{Q}$-cl $\{b\}$. Since X is pairwise R_0 we have $b \in \mathscr{P}$-cl$\{c\} \subset \mathscr{P}$-cl C and hence $\mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $C \neq \phi$ showing that $(A, C) \in \delta$. Since $(x, A) \in \delta$ iff $\mathscr{Q}$-cl$\{x\} \cap \mathscr{P}$-cl$A \neq \phi$ iff $x \in \mathscr{P}$-cl A, it follows that $\mathscr{P} = \mathscr{T}(\delta)$. Similarly $\mathscr{Q} = \mathscr{T}(\delta^{-1})$.

THEOREM 2.2. *Let*$(X, \mathscr{P}, \mathscr{Q})$ *be a pairwise* R_0 *space and let* δ_1 *be any compatible LE-proximity on X, then*

1. $(A, B) \in \delta_1$ *iff* (Q-cl A, P-cl $B) \in \delta_1$,
2. δ_1 *is coarser than the LE-proximity* δ *defined in Theorem* 3.1.

PROOF. (1) is simple. For (2) $(A, B) \in \delta \Longleftrightarrow \mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $B \neq \phi \Longrightarrow (\mathscr{Q}$-cl A, $\mathscr{P}$-cl $B) \in \delta_1 \Longleftrightarrow (A, B) \in \delta_1$.

THEOREM 2.3. *If* δ *is PR-proximity on X, then* $(X, \mathscr{T}(\delta), \mathscr{T}(\delta^{-1}))$ *is necessarily pairwise regular. Conversely a compatible PR-proximity* δ_1 *can be defined on pairwise regular space* $(X, \mathscr{P}, \mathscr{Q})$ *by setting,*

$$(A, B) \in \delta_1 \text{ iff } \mathscr{Q}\text{-cl } A \cap \mathscr{P}\text{-cl } B \neq \phi.$$

PROOF. Let $A \in \mathscr{T}(\delta)$ and $x \in A$. Then $(x, X-A) \notin \delta$ and so there exists $B \subset X$ such that $(x, X-B) \notin \delta$ and $(B, X-A) \notin \delta$ i.e.,

$$x \notin \mathscr{T}(\delta)\text{-cl } (X-B) \text{ and } \mathscr{T}(\delta^{-1})\text{-cl } B \subset A.$$

Therefore $x \in \mathscr{T}(\delta)$-int $B \subset \mathscr{T}(\delta^{-1})$-cl $B \subset A$.
Similarly if $A \in \mathscr{T}(\delta^{-1})$ and $x \in A$, then there exists $B \subset X$ such that $x \in \mathscr{T}(\delta^{-1})$-int $B \subset \mathscr{T}(\delta)$-cl $B \subset A$.

To prove that δ_1 defined above is a *PR*-proximity it is sufficient to verify (5′) only. Let $(x, A) \notin \delta_1$, so that $x \notin \mathscr{P}$-cl$A$. Let $B \subset X$ be such that $B \in \mathscr{P}$, $x \in B \subset \mathscr{Q}$-cl $B \subset X - \mathscr{P}$-cl A. Clearly $(x, X-B) \notin \delta_1$ and $(B, A) \notin \delta_1$. The compatibility follows from Theorem 2.2 and the fact that every pairwise regular space is pairwise R_0.

REMARK 2.1. Every pairwise completely regular space is pairwise regular and therefore admits of a compatible *PR*-proximity. This also follows from the fact that every quasi-proximity is a PR-proximity. But a completely regular space admits of compatible PR-proximities which are not quasi-proximities. For example on a pairwise regular space which is not pairwise normal δ_1 is a PR-proximity which is not a quasi-proximity (Theorem 3.10 of [2]). One more such PR-proximity is constructed in Theorem 2.7.

REMARK 2.2. Every pairwise regular space is pairwise R_0 and therefore admits of a compatible LE-proximity. This also follows from the fact that on a pairwise regular space δ_1 defined in Theorem 2.3 is an LE-proximity.

We now show the existence of a PR-proximity on a pairwise regular space which is not an LE-proximity.

THEOREM 2.4. *On a pairwise regular space* $(X, \mathscr{P}, \mathscr{Q})$, $(A, B) \notin \delta_2$ *iff* $\exists U \in \mathscr{P}$, $V \in \mathscr{Q} : A \subset U, B \subset V$ *and* $U \cap V = \phi$ *defines a compatible PR-proximity.*

PROOF. Again it is sufficient to verify the axiom 5′ only. Let $(x, A) \notin \delta_2$ and let $U \in \mathscr{P}$, $V \in \mathscr{Q}$ be such that $x \in U$, $A \subset V$ and $U \cap V = \phi$. Let $U_1 \in \mathscr{P}$ and $V_1 \in \mathscr{Q}$ be such that $x \in U_1$, $X - U \subset V_1$ and $U_1 \cap V_1 = \phi$. Clearly $(x, X-U) \notin \delta 2$ and since $U \in \mathscr{P}$ is such that $U \cap V = \phi$, $A \subset V \in \mathscr{Q}$, $(U, A) \notin \delta_2$. For compatibility part let $x \in A \in \mathscr{T}(\delta_2)$. Then $(x, X-A) \notin \delta_2$ and therefore there are sets $U \in \mathscr{P}, V \in \mathscr{Q}$ such that $x \in U \subset X - V \subset A$. Therefore $\mathscr{T}(\delta_2) \subset P$. If $A \in \mathscr{P}$ and $x \in A$, then by pairwise regularity we can separate x and $X-A$ by $\mathscr{P}$ and $\mathscr{Q}$ -open sets and therefore $(x, X-A) \notin \delta_2$.

REMARK 2.3. Let $(X, \mathscr{P}, \mathscr{Q})$ be a bitopological space. If δ is a compatible quasi-proximity or LE-proximity then we have

$$(A, B) \in \delta \text{ iff } (\mathscr{Q}\text{-cl}A, \mathscr{P}\text{-cl}B) \in \delta.$$

But this is not the case if δ is a PR-proximity.

Let X be an infnite set and p be a fixed point of X. let $\mathscr{P} = \{U : X-U$ is finite or $p \notin U\}$. Then $\mathscr{P}$ is a regular topology on X. Consider the pairwise regular space $(X, \mathscr{P}, \mathscr{P})$. Let A and B infinite sets such that $X-A$ and $X-B$ are not finite, $A \cap B = \phi$ and $p \notin A, p \notin B$. Then A and B are open and so $(A, B) \notin \delta_2$ the PR-proximity of Theorem 2.4. Also $\mathscr{P}\text{-cl}A = A \cup \{p\}$, $\mathscr{P}\text{-cl}B = B \cup \{p\}$ and so $(\mathscr{P}\text{-cl } A, \mathscr{P}\text{-cl } B) \in \delta_2$. Here δ_2 is not an LE-proximity. For if it were, the compatibility condition would imply $(A, B) \in \delta_2 \Longleftrightarrow (\mathscr{P}\text{-cl}A, \mathscr{P}\text{-cl}B) \in \delta_2$.

In following theorems we discuss the relations between δ_1 and δ_2.

DEFINITION. A bitopological space $(X, \mathscr{P}, \mathscr{Q})$ is said to be *pairwise extremally disconnected* if $\mathscr{Q}$-closure of each $\mathscr{P}$-open set is $\mathscr{P}$-open and $\mathscr{P}$-closure of each $\mathscr{Q}$-open set is $\mathscr{Q}$-open.

The following Theorem can be proved easily.

THEOREM 2.5. *A space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise extremally disconnected iff $A \cap B = \phi$, where A is $\mathscr{P}$-open, B is $\mathscr{Q}$-open implies $\mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $B = \phi$.*

THEOREM 2.6. Let $(X, \mathscr{P}, \mathscr{Q})$ be a pairwise regular space and let δ_1 and δ_2 be defined as in Theorems 2.4 and 2.5 respectively. Then

1. $\delta_1 \geq \delta_2$ iff $(X, \mathscr{P}, \mathscr{Q})$ is pairewise extremally disconnected,
2. $\delta_2 \geq \delta_1$ iff $(X, \mathscr{P}, \mathscr{Q})$ is pairwise normal.

PROOF. 1. Let X be pairwise extremally disconnected and that $(A, B) \notin \delta_2$. By the definition of δ_2 there exists $U \in \mathscr{P}$, $V \in \mathscr{Q}$ such that $A \subset U \subset X - V \subset X - B$ which implies $\mathscr{Q}$-cl $A \subset \mathscr{Q}$-cl U, $\mathscr{P}$-cl $B \subset \mathscr{P}$-cl V and $\mathscr{Q}$-cl $U \cap \mathscr{P}$-cl $V = \phi$, since $U \cap V = \phi$ and X is pairwise extremally disconnected. Clearly $(A, B) \notin \delta_1$. Conversely let $\delta_1 \geq \delta_2$ and let A and B be sets such that $A \in \mathscr{Q}, B \in \mathscr{P}$ and $A \cap B = \phi$. By the definition of δ_2, $(B, A) \notin \delta_2$ and therefore by the hypothesis $(B, A) \notin \delta_1$ i.e., $\mathscr{Q}$-cl $B \cap \mathscr{P}$-cl $A = \phi$ and so X is pairwise extremally disconnected.

2. Let X be pairwise normal and that $(A, B) \notin \delta_1$. By the definition of δ_1, $\mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $B = \phi$ and by the hypothesis there are sets $U \in \mathscr{P}, V \in \mathscr{Q}$ such that $\mathscr{Q}$-cl $A \subset U$, $\mathscr{P}$-cl $B \subset V$ and $U \cap V = \phi$. Therefore $(A, B) \notin \delta_2$, i.e., $\delta_2 \geq \delta_1$. Conversely suppose $\delta_2 \geq \delta_1$ and that $A = \mathscr{Q}$-cl A, $B = \mathscr{P}$-cl B, $A \cap B = \phi$. Clearly $(A, B) \notin \delta_1$ and by the hypothesis $(A, B) \notin \delta_2$, i.e., there exist sets U and V such that $A \subset U \in \mathscr{P}$, $B \subset V \in \mathscr{Q}$, $U \cap V = \phi$.

COROLLARY 2.1. *A pairwise regular space is pairwise extremally disconnected, pairwise normal iff $\delta_1 = \delta_2$.*

COROLLARY 2.2. *A pairwise regular pairwise connected space admits of at least two distinct compatible PR-proximities which are comparable iff the space is pairwise normal.*

COROLLARY 2.3. *A pairwise regular space which is not pairwise normal admits of at least two distinct compatible PR-proximities which are comparable iff the space is pairwise extremally disconnected.*

On a pairwise completely regular space we have a *PR*-proximity which is not a quasi-proximity.

THEOREM 2.7. *Let $(X, \mathscr{P}, \mathscr{Q})$ be a pairwise completely regular space. Define $(A, B) \notin \delta_3$ iff there is a $\mathscr{P}$-cozero set U and a $\mathscr{Q}$-cozero set V such that $A \subset U$, $B \subset V$, $U \cap V = \phi$, then δ_3 is a compatible PR-proximity on X.*

PROOF. We need to verify the axiom 5′ only. Let $(x, A) \notin \delta_3$ and let U, V be disjoint cozero sets containing x and A respectively. Since $\mathscr{P}$-cozero sets form a base for $\mathscr{P}$-open sets, $x \notin X-U \Longrightarrow$ there exist a $\mathscr{P}$-cozero set R and a $\mathscr{Q}$-cozero set S such that $x \in R$, $X-U \subset S$ and $R \cap S = \phi$. Clearly $(x, X-U) \notin \delta_3$ and $(U, A) \notin \delta_3$.

To see that $\mathscr{T}(\delta_3) = \mathscr{P}$, let $A \in \mathscr{T}(\delta_3)$ and $x \in A$, then $(x, X-A) \notin \delta_3$ and so there exist a $\mathscr{P}$-cozero set U and a $\mathscr{Q}$-cozero set V such that $x \in U \subset X-V \subset A$, i.e., $A \in \mathscr{P}$. Conversely if $A \in \mathscr{P}$ and $x \in A$ then there exists a $\mathscr{P}$-usc, $\mathscr{Q}$-lsc function $f: X \longrightarrow [0,1]$ such that $f(x)=0$, $f(X-A)=1$, i.e., $x \in \left\{x \in X : f(x) < \frac{1}{2}\right\}$ and $X-A \subset \left\{x \in X : \frac{1}{2} < f(x)\right\}$. But $\left\{x \in X : f(x) < \frac{1}{2}\right\}$ is $\mathscr{P}$-cozero set which is disjoint from $\left\{x \in X : f(x) > \frac{1}{2}\right\}$ a $\mathscr{Q}$-cozero set. Therefore $(x, X-A) \notin \delta_3$ i.e., $A \in \mathscr{T}(\delta_3)$. Thus $\mathscr{T}(\delta_3) = \mathscr{P}$. Similarly $\mathscr{T}(\delta_3') = \mathscr{Q}$.

DEFINITION. A bitopological space $(X, \mathscr{P}, \mathscr{Q})$, is said to be pariwise *basically disconnected* if $\mathscr{Q}$-closure of each $\mathscr{P}$-cozero set is $\mathscr{P}$-open and $\mathscr{P}$-closure of each $\mathscr{Q}$-cozero set is $\mathscr{Q}$-open.

Following theorem can be proved in a similar manner as Theorem 2.5.

THEOREM 2.8. *A space $(X, \mathscr{P}, \mathscr{Q})$ is pairwise basically disconnected iff $A \cap B = \phi$, where A is $\mathscr{P}$-cozero set, B is $\mathscr{Q}$-cozero set, implies $\mathscr{Q}$-cl $A \cap \mathscr{P}$-cl $B = \phi$.*

Thus a pairwise completely regular space admits of at least three compatible PR-proximities δ_1, δ_2 and δ_3.

The proof of the following is similar to the proof of Theorem 2.6.

THEOREM 2.9. *Let $(X, \mathscr{P}, \mathscr{Q})$ be a pairwise completely regular space and let δ_1 and δ_2 be defined as in Theorems 2.4 and 2.8. Then*

1. *$\delta_1 \geq \delta_3$ iff X is pairwise basically disconnected,*
?. *$\delta_3 \geq \delta_1$ iff X is pairwise normal.*

COROLLARY 2.4. *For a pairwise completely regular space X, the following are equivalent.*

(i) *X is pairwise normal*,

(ii) $\delta_3 \geq \delta_1$,

(iii) $\delta_2 \geq \delta_1$.

REMARK 2.4. On a pairwise completely regular space $(X, \mathscr{P}, \mathscr{Q})$, let δ_4 denote the PR-(in fact quasi-) proximity defined by $(A, B) \notin \delta_4$ iff there exists a map f which is $\mathscr{P}$-usc, $\mathscr{Q}$-lsc such that f is zero on A and 1 on B. Thus on a pairwise completely regular space we have four compatible PR-proximities. In general $\delta_2 \geq \delta_3 \geq \delta_4$. If the space is pairwise normal, then $\delta_2 \geq \delta_4 \geq \delta_1$ and $\delta_3 \geq \delta_1$. If it is pairwise extremally disconnected, then $\delta_1 \geq \delta_2$ and on a pairwise basically disconnected space $\delta_1 \geq \delta_3$. On a pairwise normal bi T_1-space $\delta_1 = \delta_4$ and $\delta_2 \geq \delta_1$. Therefore on an pairwise extremally disconnected pairwise normal bi T_1-space $\delta_1 = \delta_2 = \delta_3 = \delta_4$.

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