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CONNECTIVITY PROPERTIES OF BITOPOLOGICAL HYPERSPACES

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1. Introduction

A bitopological space is a triple $(X, \mathscr{T}_1, \mathscr{T}_2)$ where $\mathscr{T}_1$ and $\mathscr{T}_2$ are arbitrary topologies on X. Kelly [4] initiated the systematic study of such spaces and several authors have contributed to the subsequent development of various bitopological properties. The problem of defining bitopological connectedness and related concepts has been considered, in particular, by Pervin [9], Swart [11], Dutta [2], Reilly [10] and Birsan [1]. They introduced and studied pairwise connectedness, pairwise totally disconnectedness pairwise zero-dimensional and pairwise local connectedness respectively for bitopological spaces.

The authors have introduced in [12] the notion of bitopological hyperspaces. For any bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ we have considered the natural bitopological hyperspaces $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ and $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ where 2^X is the collection of all non-empty $\mathscr{T}_1$-closed or $\mathscr{T}_2$-closed sets including the singletons, $C(X)$ is the collection of all elements of 2^X which are bicompact, $2^{\mathscr{T}_1}$ and $2^{\mathscr{T}_2}$ are finite topologies induced by $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively on 2^X or $C(X)$ as defined in [6]. In [12] we observed that various separation axioms for $(X, \mathscr{T}_1, \mathscr{T}_2)$ are weakened to $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ and are equivalent to $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$. Michael [8] has shown that connectedness, total disconnectedness, zero-dimensional and local connectedness of a topological space are carried over to hyperspaces. In the present paper we study bitopological analogue of these concepts. We show that pairwise connectedness of $(X, \mathscr{T}_1, \mathscr{T}_2)$ is equivalent to that of $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ whereas pairwise total disconnectedness pairwise zero-dimensionality and pairwise local connectedness of $(X, \mathscr{T}_1, \mathscr{T}_2)$ are equivalent to that of $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$. Also we prove some other related results.

2. Notations and definitions

In this section we define some notations and terms which we have used in this

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paper.

NOTATION 2.1. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be any bitopological space. Define:

$$\mathfrak{A}(X)=\{A\subset X : A\neq\phi\},$$
$$2^X=\{A\subset X : A\neq\phi, A \text{ is } \mathscr{T}_1\text{-closed or } \mathscr{T}_2\text{-closed}\},$$
$$C(X)=\{A\in 2^X : A \text{ is } \mathscr{T}_1 \text{ as well as } \mathscr{T}_2 \text{ compact}\},$$
$$\mathscr{F}(X)=\{A\in 2^X : A \text{ is finite}\},$$
$$\mathscr{F}_n(X)=\{A\in 2^X : A \text{ contains at most } n\text{-elements}\}.$$

NOTATION 2.2. For any arbitrary sets G and H, define:

$$B(G)=\{F\in 2^X : F\subset G\} \text{ and } C(H)=\{F\in 2^X : F\cap H\neq\phi\}.$$

NOTATION 2.3. If $A_0, A_1, A_2, \cdots, A_n$ is any system of arbitrary sets, define:

$$B(A_0, A_1, A_2, \cdots, A_n)=\{F\in 2^X : F\subset A_0, F\cap A_i\neq\phi \text{ for } i\leq n\}$$
$$=B(A_0)\cap C(A_1)\cap\cdots\cap C(A_n).$$

Without loss of generality, it can be assumed that $A_i\subset A_0$ for $i=1, 2, \cdots, n$ since

$$B(A_0, A_1, A_2, \cdots, A_n)=B(A_0, A_1\cap A_0, \cdots, A_n\cap A_0).$$

DEFINITION 2.1. For any topological space $(X, \mathscr{T})$, the totality of sets $B(G)$ and $C(H)$ forms a subbase for a topology $2^{\mathscr{T}}$ on 2^X provided G and H run over $\mathscr{T}$. Also the collection of all $B(G_0, G_1, \cdots, G_n)$ forms a basis for the same topology $2^{\mathscr{T}}$ on 2^X where $G_0, G_1, \cdots, G_n$ are a finite collection of members of $\mathscr{T}$. The topology $2^{\mathscr{T}}$ is called the *finite topology* [8] or *the exponential topology* [6].

REMARK 2.1. Michael [8] followed the following convention for defining the finite topology $2^{\mathscr{T}}$ for 2^X. For any finite collection of open sets $G_1, G_2, \cdots, G_n$ be defined; $\langle G_1, G_2, \cdots, G_n\rangle=\{F\in 2^X : F\subset \bigcup_{i=1}^{n} G_i, F\cap G_i\neq\phi \text{ for each } i\}$ then the collection $\{\langle G_1, G_2, \cdots, G_n\rangle : G_1, G_2, \cdots, G_n\in\mathscr{T}\}$ forms a basis for $2^{\mathscr{T}}$.

DEFINITION 2.2. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be any bitopological space. The collection of the form $B(G_0, G_1, G_2, \cdots, G_n)$ with $G_0, G_1, G_2, \cdots, G_n$ in $\mathscr{T}_1$ forms a basis for the finite topology $2^{\mathscr{T}_1}$ on 2^X and collection $B(H_0, H_1, H_2, \cdots, H_m)$ with $H_0, H_1, H_2, \cdots, H_m$ in $\mathscr{T}_2$ forms a basis for the finite topology $2^{\mathscr{T}_2}$ on 2^X. Thus we get a bitopological space $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ which we call a *bitopological hyperspace*.

In a similar manner we can define $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$, $(\mathfrak{A}(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ and $(\mathscr{F}(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ bitopological hyperspaces.

NOTATION 2.4. When there is no confusion cl(F) (int F) denotes the closure (interior) of F otherwise we shall write $\mathscr{T}$-cl F ($\mathscr{T}$-int F) which denotes that the closure (interior) of F is taken with respect to $\mathscr{T}$.

Next we collect various definitions used in the text:

DEFINITION 2.3. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise connected* [9] if X cannot be expressed as the union of two non-empty disjoint sets A and B such that A is $\mathscr{T}_1$-open and B is $\mathscr{T}_2$-open (Hence A is $\mathscr{T}_2$-closed, B is $\mathscr{T}_1$-closed). Otherwise $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called *pairwise disconnected* and in that case X is written as $X=A/B$.

DEFINITION 2.4. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise totally disconnected* [11] if for each pair of distinct points x, y there exists a disconnection $X=A/B$ such that $x\in A$, $y\in B$ or $x\in B$, $y\in A$.

DEFINITION 2.5. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise zero-dimensional* [2, 10] if $\mathscr{T}_1$ has a base consisting of $\mathscr{T}_1$-open $\mathscr{T}_2$-closed sets and $\mathscr{T}_2$ has a base consisting of $\mathscr{T}_2$-open $\mathscr{T}_1$-closed sets.

DEFINITION 2.6. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise locally connected* [1] if for each $x\in X$ and each $\mathscr{T}_i$-open set G containing x there is a pairwise connected $\mathscr{T}_i$-open set H satisfying $x\in H\subset G$, $i\neq j$, $i, j=1, 2$.

DEFINITION 2.7. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ and $(Y, \mathscr{T}_1^*, \mathscr{T}_2^*)$ be any bitopological spaces. A function $f:(X, \mathscr{T}_1, \mathscr{T}_2)\longrightarrow(Y, \mathscr{T}_1^*, \mathscr{T}_2^*)$ is said to be *pairwise continuous* (respectively *pairwise homeomorphism*) if the induced functions f: $(X, \mathscr{T}_1)\longrightarrow(Y, \mathscr{T}_1^*)$ and f: $(X, \mathscr{T}_2)\longrightarrow(Y, \mathscr{T}_2^*)$ are continuous (respectively homeomorphism).

3. Bitopological hyperspaces

Through out the whole paper, we assume that in $(X, \mathscr{T}_1, \mathscr{T}_2)$ one of the topologies is T_1 so that singletons are in 2^X.

For a topological space $(X, \mathscr{T})$ it is shown that if $\mathscr{B}$ is a collection of subsets of X which is a subcollection of 2^X and is connected in the finite topology $2^{\mathscr{T}}$ and one of whose elements is connected then $\bigcup_{E\in\mathscr{B}} E$ is connected [see 8, Proposition 2.8]. Here we show the bitopological analogue of this theorem:

THEOREM 3.1. *If $\mathscr{B}$ is a collection of subsets of $(X, \mathscr{T}_1, \mathscr{T}_2)$ which is a subcollection of 2^X and is pairwise connected in $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ and one of whose elements is pairwise connected then $\bigcup_{E\in\mathscr{B}} E$ is pairwise connected.*

PROOF. Let $S=\bigcup_{E\in\mathscr{B}} E$. If possible let S be not pairwise connected so that $S=M\cup N$ where $M\neq\phi$, $N\neq\phi$, M is $\mathscr{T}_1$-open, N is $\mathscr{T}_2$-open such that $M\cap N=\phi$. Given that one of the elements of say F is pairwise connected so $F\subset M$ or $F\subset N$ [see 9, Theorem E]. Without loss of generality we may assume that $F\subset M$. Now define

$$\tilde{P}=\{A\in 2^X : A\subset M\}\cap\mathscr{B},$$
$$\tilde{Q}=\{A\in 2^X : A\cap N\neq\phi\}\cap\mathscr{B}.$$

Then $\tilde{P}$ is $2^{\mathscr{T}_1}|_{\mathscr{B}}$-open, $\tilde{Q}$ is $2^{\mathscr{T}_2}|_{\mathscr{B}}$-open.

The proof will be completed if we show that $\mathscr{B}=\tilde{P}\cup\tilde{Q}$ and $\tilde{P}\cap\tilde{Q}=\phi$, for, it would imply that $\mathscr{B}$ is pairwise disconnected.

Let $A\in\mathscr{B}$ then $A\subset S$. Now either $A\subset M$ or or $A\not\subset M$. In the former case $A\in\tilde{P}$ and in the latter $A\in\tilde{Q}$ so that $\mathscr{B}=\tilde{P}\cup\tilde{Q}$.

Next, let if possible $\tilde{P}\cap\tilde{Q}\neq\phi$ then there exists an $L\in\tilde{P}\cap\tilde{Q}$ implying that $L\in\tilde{P}$ and $L\in\tilde{Q}$, that is, $L\subset M$ and $L\cap N\neq\phi$ which implies that $M\cap N\neq\phi$ a contradiction. Hence $\tilde{P}\cap\tilde{Q}=\phi$. Therefore S is pairwise connected.

DEFINITION 3.1. A bitoplogical space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise normal* [4] if for every pair of sets F_1, F_2 where F_1 is $\mathscr{T}_i$-closed, F_2 is $\mathscr{T}_j$-closed such that $F_1\cap F_2=\phi$ there exists $\mathscr{T}_j$-open set G and $\mathscr{T}_i$-open set H satisfying $F_1\subset G_1$, $F_2\subset H$, $G\cap H=\phi$, $i\neq j$, $i,j=1, 2$.

THEOREM 3.2. *The family* $\mathscr{C}\subset 2^X$ *of pairwise connected subsets of a pairwise normal space* $(X, \mathscr{T}_1, \mathscr{T}_1)$ *is closed in* $(2^X, 2^{\mathscr{T}_1\vee\mathscr{T}_2})$.

PROOF. We show that $2^X-\mathscr{C}$ is open in 2^X with respect to $2^{\mathscr{T}_1\vee\mathscr{T}_2}$. Let $A\in 2^X-\mathscr{C}$ then A is non pairwise connected $\mathscr{T}_1$-closed $\mathscr{T}_2$-closed set.

So $A=M\cup N$ where $M\neq\phi$, $\neq N$, M is $\mathscr{T}_1$-closed, N is $\mathscr{T}_2$-closed, such that $M\cap N=\phi$ Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise normal, there exist $\mathscr{T}_2$-open set G and $\mathscr{T}_1$-open set H satisfying $M\subset G$, $N\subset H$, $G\cap H=\phi$. Define:

$$\mathscr{U}=\{F\in 2^X : F\subset G\cup H : F\cap G\neq\phi\neq F\cap H\}$$

then $\mathscr{U}$ is open in 2^X with respect to $2^{\mathscr{T}_1\vee\mathscr{T}_2}$ and $A\in\mathscr{U}$. Obviously $\mathscr{U}\cap\mathscr{C}=\phi$. Hence $A\in\mathscr{U}\subset 2^X-\mathscr{C}$ so $2^X-\mathscr{C}$ is open in $2^{\mathscr{T}_1\vee\mathscr{T}_2}$ whence $\mathscr{C}$ is closed in $2^{\mathscr{T}_1\vee\mathscr{T}_2}$. Hence the theorem.

Now before we establish the next theorem we need the following Lemma:

LEMMA 3.1. *Let* $f_i : (Y, \mathscr{T}_1, \mathscr{T}_2)\longrightarrow(X, \mathscr{T}_1^*, \mathscr{T}_2^*)$ *for* $i=1, 2, \cdots, n$ *be pairwise* *continuous and* $(X, \mathcal{T}_1^*, \mathcal{T}_2^*)$ *be such that one of the topologies say* $\mathcal{T}_1^*$ *is* T_1. *Define*

$F:(Y, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (2^X, 2^{\mathcal{T}_1^*}, 2^{\mathcal{T}_2^*}) : F(y)=\{f_1(y), f_2(y), \cdots, f_n(y)\}$ *then* F *is pairwise continuous.*

PROOF. We show that $F:(Y, \mathcal{T}_1) \longrightarrow (2^X, 2^{\mathcal{T}_1^*})$ and $F:(Y, \mathcal{T}_2) \longrightarrow (2^X, 2^{\mathcal{T}_2^*})$ are continuous. Let us consider $F:(Y, \mathcal{T}_1) \longrightarrow (2^X, 2^{\mathcal{T}_1^*})$. We show the inverse image under F of any subbasic open set in 2^X is open. The subbasic open sets look like:

$B(H)=\{F \in 2^X : F \subset H\}$ and $C(H)=\{F \in 2^X : F \cap H \neq \phi\}$ where H is $\mathcal{T}_1^*$-open

Case I. Let $y_0 \in Y$ and $F(y_0) \in B(H)$ then we find a $\mathcal{T}_1$-open set Q such that $y_0 \in Q$ and $F(Q) \subset B(H)$. Now

$$
\begin{aligned}
&F(y_0) \in B(H)\\
&\Longrightarrow \{f_1(y_0), f_2(y_0), f_3(y_0) \cdots, f_n(y_0)\} \in B(H)\\
&\Longrightarrow \{f_1(y_0), f_2(y_0), f_3(y_0), \cdots, f_n(y_0)\} \subset H\\
&\Longrightarrow f_i(y_0) \in H \text{ for each } i=1, 2, \cdots, n.
\end{aligned}
$$

Since each f_i pairwise continuous there exist $\mathcal{T}_1$-open sets Q_i such that $y_0 \in Q_i$, $f_i(Q_i) \subset H$. Put $Q=\bigcap_{i=1}^{n} Q_i$ then Q is $\mathcal{T}_1$-open set such that

$$y_0 \in Q \text{ and } F(Q) \subset B(H).$$

Case II. Let $y_0 \in Y$ and $F(y_0) \in C(H)$ then we find a $\mathcal{T}_1$-open set Q such that $y_0 \in Q$ and $F(Q) \subset C(H)$. Now

$$
\begin{aligned}
&F(y_0) \in C(H)\\
&\Longrightarrow \{f_1(y_0), f_2(y_0), \cdots, f_n(y_0)\} \in C(H)\\
&\Longrightarrow \{f_1(y_0), f_2(y_0), \cdots, f_n(y_0)\} \cap H \neq \phi\\
&\Longrightarrow f_i(y_0) \in H \text{ for some } i=1, 2, \cdots, n.
\end{aligned}
$$

Since f_i is pairwise continuous, there exists a $\mathcal{T}_1$-open set Q_i such that $y_0 \in Q_i$ and $f_i(Q_i) \subset H$. Put $Q=Q_i$ then we have $y_0 \in Q$ and $F(Q) \subset C(H)$.

Hence $F:(Y, \mathcal{T}_1) \longrightarrow (2^X, 2^{\mathcal{T}_1^*})$ is continuous. Similarly it can be shown that $F:(Y, \mathcal{T}_2) \longrightarrow (2^X, 2^{\mathcal{T}_2^*})$ is continuous. Hence $F:(Y, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (2^X, 2^{\mathcal{T}_1^*}, 2^{\mathcal{T}_2^*})$ is pairwise continuous.

THEOREM 3.3. *If* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *is pairwise connected space then* $(\mathcal{F}_n(X), 2^{\mathcal{T}_1},$

$2^{\mathscr{T}_2}$) *is pairwise connected.*

PROOF. In the Lemma 3.1, put $Y=X^n$ and define $f_i : X^n \longrightarrow X : f_i(x_1, x_2, \cdots, x_n)=x_i$ then f_i is pairwise continuous mapping. Now define

$$F : X^n \longrightarrow \mathscr{F}_n(X) : F(x_1, x_2, \cdots, x_n)=\{f_i(x_1, x_2, \cdots, x_n) : i=1, 2, \cdots, n\},$$

then F is pairwise continuous and onto mapping. As $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise connected so is $(\mathscr{F}_n(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ since the pairwise continuous onto image of a pairwise connected space is pairwise connected.

Next we mention the following result from [6] which we use very oftenly :

LEMMA 3.2. (X) *is dense in* $2^X/\mathfrak{A}(X)/C(X)$.

PROOF. See [6, Theorem 4 : p. 163]

THEOREM 3.4. *Let* $\mathscr{F}(X) \subset \mathscr{G} \subset \mathfrak{A}(X)$. *If one of the spaces* X, $\mathscr{F}_n(X)$ *or* $\mathscr{G}$ *is pairwise connected then all of them are pairwise connected.*

PROOF. Suppose that X is pairwise connected then by Theorem 3.3, $\mathscr{F}_n(X)$ is pairwise connected. In order to show that $\mathscr{F}(X)$ is pairwise connected it is sufficient to show that every pair of distinct elements of $\mathscr{F}(X)$ belongs to some pairwise connected set contained in $\mathscr{F}(X)$ [see 9, Theorem E, Corollary 1].

Let $M, N \in \mathscr{F}(X)$ then they both belong to $\mathscr{F}_n(X)$ for some n which is pairwise connected and is obviously contained in $\mathscr{F}(X)$. Hence $\mathscr{F}(X)$ is pairwise connected. Also by Lemma 3.2, we have,

$$\mathscr{F}(X) \subset \mathscr{G} \subset \mathfrak{A}(X)=2^{\mathscr{T}_1}\text{-cl } \mathscr{F}(X) \cap 2^{\mathscr{T}_2}\text{-cl } \mathscr{F}(X),$$

which implies that G is also pairwise connected.

Now suppose that $\mathscr{F}_n(X)$ is pairwise connected. Since singletons belong to $\mathscr{F}_n(X)$ which are pairwise connected, by Theorem 3.1, $X=\bigcup_{F \in \mathscr{F}_n(X)} F$ is pairwise connected.

By a similar argument it follows that if $\mathscr{G}$ is pairwise connected then X is pairwise connected. Hence the theorem.

Now we prove the following equivalence:

THEOREM 3.5. $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is pairwise connected if and only if* $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ *is pairwise connected.*

PROOF. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be pairwise connected. Since $\mathscr{F}(X) \subset 2^X \subset 2^{\mathscr{T}_1}\text{-cl } \mathscr{F}(X) \cap 2^{\mathscr{T}_2}\text{-cl } \mathscr{F}(X)$, it follows from Theorem 3.4 that 2^X is pairwise connected.

Conversely, let $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ be pairwise connected and let if possible $(X, \mathscr{T}_1, \mathscr{T}_2)$ be pairwise disconnected then $X=G\cup H$ where $G\neq\phi, H\neq\phi$, G is $\mathscr{T}_1$-open, H is $\mathscr{T}_2$-open such that $G\cap H=\phi$. Put $\mathscr{U}=B(G)$ and $\mathscr{V}=C(H)$ then $\mathscr{U}$ is $2^{\mathscr{T}_1}$-open $\mathscr{V}$ is $2^{\mathscr{T}_2}$-open such that $2^X=\mathscr{U}\cup\mathscr{V}$ and $\mathscr{U}\cap\mathscr{V}=\phi$ which contradicts the fact that $(2^X, 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ is pairwise connected. Hence $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise connected.

THEOREM 3.6. *If* $A_1, A_2, \cdots, A_n$ *are pairwise connected subsets of* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *and if* $\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle\subset\mathscr{G}\subset 2^{\mathscr{T}_1}\text{-cl}\langle A_1, A_2, \cdots, A_n\rangle\cap 2^{\mathscr{T}_2}\text{-cl}\langle A_1, A_2, \cdots, A_n\rangle$ *then* $\mathscr{G}$ *is pairwise connected.*

PROOF. Since each A_i is pairwise connected, by Theorem 3.4, so is each $\mathscr{F}(A_i)$ and hence $\mathscr{F}(A_1)\times\mathscr{F}(A_2)\times\cdots\times\mathscr{F}(A_n)$ is pairwise connected in $[\mathscr{A}(X)]^n$. Now define a mapping:

$$\sigma : \mathscr{F}(A_1)\times\mathscr{F}(A_2)\times\cdots\times\mathscr{F}(A_n)\longrightarrow\mathscr{A}(X) \,/\, \sigma(E_1, E_2, \cdots, E_n)=\bigcup_{i=1}^{n} E_i$$

then σ is pairwise continuous [8, proposition 5.8, 5.8.1]. But $\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle$ is the pairwise continuous onto image of σ and hence is also pairwise connected. Therefore by [9, Theorem E, corollary 3] $2^{\mathscr{T}_1}\text{-cl}\ (\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)\cap 2^{\mathscr{T}_2}\text{-cl}\ (\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)$ is pairwise connected.

Now it can easily be verified that

$$2^{\mathscr{T}_1}\text{-cl}(\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)\cap 2^{\mathscr{T}_2}\text{-cl}\ (\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)$$
$$=2^{\mathscr{T}_1}\text{-cl}\langle A_1, A_2, \cdots, A_n\rangle\cap 2^{\mathscr{T}_2}\text{-cl}\langle A_1, A_2, \cdots, A_n\rangle$$

so we have,

$\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle\subset\mathscr{G}\subset 2^{\mathscr{T}_1}\text{-cl}\ \langle A_1, A_2, \cdots, A_n\rangle\cap 2^{\mathscr{T}_2}\text{-cl}\ \langle A_1, A_2, \cdots, A_n\rangle=2^{\mathscr{T}_1}\text{-cl}(\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)\cap 2^{\mathscr{T}_2}\text{-cl}(\mathscr{F}(X)\cap\langle A_1, A_2, \cdots, A_n\rangle)$ implies that $\mathscr{G}$ is pairwise connected.

COROLLARY 3.1. *If* $A_1, A_2, A_3, \cdots, A_n$ *are pairwise connected subsets of* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *then* $B(A_0, A_1, A_2, \cdots, A_n)$ *is also pairwise connected where* $A_0=\bigcup_{i=1}^{n} A_i$.

PROOF. Taking for $\mathscr{G}=\langle A_1, A_2, \cdots, A_n\rangle$ in Theorem 3.6, it follows that $\langle A_1, A_2, \cdots, A_n\rangle$ is pairwise connected. But $\langle A_1, A_2, \cdots, A_n\rangle=B(A_0, A_1,$

$A_2, \cdots, A_n)$ where $A_0=\bigcup_{i=1}^{n} A_i$ whence $B(A_0, A_1, A_2 \cdots, A_n)$ is pairwise connected.

The following theorem shows the equivalence of pairwise local connectedness of $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(C(X), 2^{\mathcal{T}_1}, 2^{\mathcal{T}_2})$.

THEOREM 3.7. *Let $\mathcal{F}(X)\subset\mathcal{G}\subset C(X)$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise local connected if and only if $\mathcal{G}$ is pairwise local connected.*

PROOF. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be pairwise local connected. Let $E\in C(X)$ and let $\mathcal{U}$ be any $2^{\mathcal{T}_i}$-open neighbourhood of E in $\mathcal{A}(X)$ so we can find a $2^{\mathcal{T}_i}$-basic open set $B(G_0, G_1, \cdots, G_n)$ with $G_0, G_1, G_2, \cdots, G_n$ in $\mathcal{T}_i$ such that

$$E\in B(G_0, G_1, \cdots, G_n).$$

$E\in B(G_0, G_1, G_2, \cdots, G_n)$ implies that $E\subset G_0$ and $E\cap G_i\neq\phi$ for each $i=1, 2, \cdots, n$. $E\subset G_0$ implies that G_0 is $\mathcal{T}_i$-open neighbourhood of each $x\in E$. Since $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise local connected for each $x\in E$ we can find a $\mathcal{T}_i$-open pairwise connected set V_x such that

$$x\in V_x\subset G_i.$$

Now $\mathcal{A}=\{V_x : x\in E\}$ is a $\mathcal{T}_i$-open covering of E which is $\mathcal{T}_i$-compact so it admits of a finite subcovering say $\{V_1, V_2, \cdots, V_m\}$.

Also $E\cap G_i\neq\phi$ for each $i=1, 2, \cdots, n$ let $x_i\in E\cap G_i$ for each i. Then G_i is a $\mathcal{T}_i$-open neighbourhood of x_i so by pairwise local connectedness of $(X, \mathcal{T}_1, \mathcal{T}_2)$ we can find a $\mathcal{T}_i$-open pairwise connected set H_i such that

$$x_i\in H_i\subset G_i \text{ for each } i=1, 2, \cdots, n.$$

Thus we get a finite collection of $\mathcal{T}_i$-open pairwise connected sets $\{V_1, V_2, \cdots, V_m, H_1, H_2, \cdots, H_n\}$. Let

$$V_0=\bigcup_{i=1}^{m} V_i \cup \bigcup_{j=1}^{n} H_j.$$

Put $\mathcal{V}=B(V_0, V_1, V_2, \cdots, V_m, H_1, H_2, \cdots, H_n)$ then by Corollary 3.1, $\mathcal{V}$ is pairwise connected and $2^{\mathcal{T}_i}$-open. Obviousiy, we have

$$E\in\mathcal{V}\subset B(G_0, G_1, G_2, \cdots, G_n).$$

Hence $C(X)$ is pairwise locally connected. Also by Theorem 3.6, $\mathcal{G}$ is pairwise locally connected.

Conversely, let $\mathcal{G}$ be pairwise locally connected and let $x\in X$. Let H be any $\mathcal{T}_i$-open neighbourhood of x. Then $\{x\}\in C(X)$ and $B(H)$ is $2^{\mathcal{T}_i}$-open neighbourhood of $\{x\}$ so by hypothesis, there exists a $2^{\mathscr{T}_i}$-open pairwise connected neighbourhood say $\mathscr{V}$ of $\{x\}$ in $\mathscr{G}$ such that

$$\{x\}\in\mathscr{V}\subset B(H).$$

Then we have

$\mathscr{V}=\bigcup_{\lambda} B_\lambda(G_{0_\lambda}, G_{1_\lambda}, \cdots, G_{n_\lambda})$ where $B_\lambda(G_{0_\lambda}, G_{1_\lambda}, \cdots G_{n_\lambda})$ are $2^{\mathscr{T}_i}$-basic open sets. Define

$$B_\lambda=\bigcup_{i=0}^{n_\lambda} G_{i_\lambda}.$$

Put $V=\bigcup_{\lambda} B_\lambda$ then V is $\mathscr{T}_i$-open neighbourhood of x and $V\subset H$. By Theorem 3.1, V is pairwise connected since $\{x\}\in\mathscr{V}$ which is pairwise connected. Hence the Theorem.

NOTATION 3.1. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space. Define $\check{C}(X)=\{A\subset X:A$ is $\mathscr{T}_1$-compact$\}$. Consider $(\check{C}(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ where $2^{\mathscr{T}_1}$ and $2^{\mathscr{T}_2}$ are finite topologies on $\check{C}(X)$ induced by $\mathscr{T}_1$ and $\mathscr{T}_1$ respectively.

Now we prove the following equivalence:

THEOREM 3.8. *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise totally disconnected if and only if $(\check{C}(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ is so.*

PROOF. Suppose that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise totally disconnected. Let $F_1, F_2 \in C(X)$ such $F_1\neq F_2$. Then $(F_1-F_2)\cup(F_2-F_1)\neq\phi$. Without loss of generality we may assume that $F_1-F_2\neq\phi$ so there exists an $x_0\in F_1$ such that $x_0\notin F_2$. $x_0\notin F_2$ implies that $x_0\neq y$ for each $y\in F_2$. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise totally disconnected we can find a separation of X such that $X=U_y/V_{yx_0}$ with $x_0\in V_{yx_0}$ $y\in U_y$, U_y is $\mathscr{T}_1$-open and V_{yx_0} is $\mathscr{T}_2$-open.

Now, $\mathscr{A}=\{U_y : y\in F_2\}$ is a $\mathscr{T}_1$-open cover of F_2 which is $\mathscr{T}_1$-compact so it admits of a finite subcover say $\{U_{y_i}\ i=1, 2, \cdots, n\}$. Now corresponding to each i we have $V_{y_ix_0}$ such that $X=U_{y_i}/V_{y_ix_0}$.

Put $U=\bigcup_{i=1}^{n} U_{y_i}$, $V=\bigcap_{i=1}^{n} V_{y_ix_0}$, then U is $\mathscr{T}_1$-open and V is $\mathscr{T}_2$-open set. Consider now $\mathscr{U}=B(U)$ and $\mathscr{V}=B(X,V)$ then $\mathscr{U}$ is $2^{\mathscr{T}_1}$-open $\mathscr{V}$ is $2^{\mathscr{T}_2}$-open such that $F_2 \in\mathscr{U}$, $F_1\in\mathscr{V}$, $C(X)=\mathscr{U}\cup\mathscr{V}$ and $\mathscr{U}\cap\mathscr{V}=\phi$. Hence $(\check{C}(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ is pairwise totally disconnected.

Conversely, let $x, y\in X$ such $x\neq y$. Then $\{x\}$, $\{y\}\in C(X)$ and are distinct. Since

$\check{C}(X)$, $2^{\mathcal{T}_1}$, $2^{\mathcal{T}_2}$) is pairwise totally disconnected, there exists a separation of $\check{C}(X)$ say $\mathcal{U}$ and $\mathcal{V}$ such that $\{x\}\in\mathcal{U}$, $\{y\}\in\mathcal{V}$ where $\mathcal{U}$ is $2^{\mathcal{T}_1}$-open and $\mathcal{V}$ is $2^{\mathcal{T}_2}$-open. We have

$$\mathcal{U}=\bigcup_{\alpha} B_{\alpha}(G_0^{\alpha},\ G_1^{\alpha},\ \cdots,\ G_{n_\alpha}^{\alpha}),$$

and

$$\mathcal{V}=\bigcup_{\beta} B_{\beta}(H_0^{\beta},\ H_1^{\beta},\ \cdots,\ H_{m_\beta}^{\beta}),$$

where

$$B_{\alpha}(G_0^{\alpha},\ G_1^{\alpha},\ \cdots,\ G_{n_\alpha}^{\alpha}) \text{ and } B_{\beta}(H_0^{\beta},\ H_1^{\beta},\ \cdots,\ H_{m_\beta}^{\beta}),$$

are $2^{\mathcal{T}_1}$, and $2^{\mathcal{T}_2}$ basic open sets respectively.

Put $G_\alpha=\bigcap_{i=1}^{n_\alpha} G_i$ and $H=\bigcap_{j=1}^{m_\beta} H_j$ then G and H are $\mathcal{T}_1$-open and $\mathcal{T}_2$-open respectively. Define $G=\bigcup_{\alpha} G_\alpha$ and $H=\bigcup_{\beta} H_\beta$ then $x\in G$, $y\in H$ and $X=G\cup H$ such that $G\cap H=\phi$ where G is $\mathcal{T}_1$-open and H is $\mathcal{T}_2$-open. Hence the theorem.

COROLLARY 3.2. *(X, $\mathcal{T}_1$, $\mathcal{T}_2$) is pairwise totally disconnected if and only if ($C(X)$, $2^{\mathcal{T}_1}$, $2^{\mathcal{T}_2}$) is pairwise totally disconnected.*

COROLLARY 3.3. *If (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is K-compact* [5] *then (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is pairwise totally disconnected if and only if (2^X, $2^{\mathcal{T}_1}$, $2^{\mathcal{T}_2}$) is pairwise totally disconnected.*

PROOF. It follows from the fact that in a K-compact bitopological space (X, $\mathcal{T}_1$, $\mathcal{T}_2$) every proper $\mathcal{T}_i$-closed set is $\mathcal{T}_j$-compact, $i\neq j, i, j=1,\ 2$.

COROLLARY 3.4. *If (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is K-compact and 2^X denotes the collection of non-empty closed subsets with respect to $\mathcal{T}_1$ (respectively $\mathcal{T}_2$) topology then (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is pairwise totally disconnected if and only if (2^X, $2^{\mathcal{T}_1}$, $2^{\mathcal{T}_2}$) is pairwise totally disconnected.*

PROOF. Trivial.

COROLLARY 3.5. *If (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is FHP-compact* [3] *then (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is pairwise totally disconnected if and only if (2^X, $2^{\mathcal{T}_1}$, $2^{\mathcal{T}_2}$) is pairwise totally disconnected.*

In [8] it is shown that the mapping $i:(X,\mathcal{T})\longrightarrow(C(X),\ 2^{\mathcal{T}})$ $i(x)=x$ is a homeomorphism. The same mapping $i:(X,\ \mathcal{T}_1,\ \mathcal{T}_2)\longrightarrow(C(X),\ 2^{\mathcal{T}_1},\ 2^{\mathcal{T}_2})$ is therefore, pairwise homeomorphism. We make use of this fact in the following theorem:

THEOREM 3.9. *A bitopological space* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is pairwise zero-dimensional if and only if* $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ *is is pairwise zero-dimensional.*

PROOF. $(X, \mathscr{T}_1, \mathscr{T}_2,)$ be pairwise zero-dimensional. Let $A \in C(X)$ and $\mathscr{U}$ be any $2^{\mathscr{T}_i}$-open neighbourhood of A. Then we can find a $2^{\mathscr{T}_i}$ basic open set $\mathscr{V}$ such that

$$A \in \mathscr{V} \subset \mathscr{U}$$

so there exists a system of $\mathscr{T}_i$-open sets $G_0, G_1, G_2, \cdots, G_n$ such that,

$$\mathscr{V} = B(G_0, G_1, G_2, \cdots, G_n).$$

Now $A \in B(G_0, G_1, G_2, \cdots, G_n)$ implies that $A \subset G_0$, $A \cap G_i \neq \phi$ for each $i=1, 2, \cdots, n$. Since A is bi-compact, G_0 is $\mathscr{T}_i$-open neighbourhood of A and $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise zero-dimensional we can find a $\mathscr{T}_i$-open $\mathscr{T}_j$-closed set H_0 such that

$$A \subset H_0 \subset G_0.$$

Also $A \cap G_i \neq \phi$ for each $i=1, 2, \cdots, n$. Let $x_i \in A \cap G_i$ so that G_i is a $\mathscr{T}_i$-open neighbourhood of x_i hence we can find a $\mathscr{T}_i$-open $\mathscr{T}_j$-closed set H_i such that

$$x_i \in H_i \subset G_i.$$

Put $\mathscr{W} = B(H_0, H_1, H_2, \cdots, H_n)$ then $\mathscr{W}$ is $2^{\mathscr{T}_i}$-open $2^{\mathscr{T}_j}$-closed such that $A \in \mathscr{W} \subset \mathscr{U}$. Hence $(C(X), 2^{\mathscr{T}_1}, 2^{\mathscr{T}_2})$ is pairwise zero-dimensional.

Converse follows from the fact that X is pairwise homeomorphic to a subcollection of $C(X)$ since pairwise zero-dimensionality is a bitopological property and it is hereditary too.

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