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ON DIAZ-METCALF'S COMPLEMENTARY TRIANGLE INEQUALITY

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1. Introduction

Let E be a vector space over the field K of real or complex numbers. Nath [3] defines a generalized semi-inner product (abbreviated to g.s.i.p.) on E as a functional $[x,y]$ on $E\times E$ with the following properties:

(1) $[x+y,z]=[x,z]+[y,z]$, $[\lambda x,y]=\lambda[x,y]$, for all x,y,z in E and λ in K,

(2) $[x,x]>0$ for $x\neq 0$,

(3) $|[x,y]|\leq [x,x]^{1/p}[y,y]^{(p-1)/p}$, $1<p<\infty$.

E, equipped with a g.s.i.p., is called a g.s.i.p. space, and a g.s.i.p. induces a norm on E by setting $\|x\|=[x,x]^{1/p}$. A normed vector space can be made into a g.s.i.p. space, as has been shown in [3].

If we put $p=2$ in the above definition, we get semi-inner product of Lumer [2].

In the present note, we extend Complementary Triangle Inequality of Diaz and Metcalf [1] to g.s.i.p. spaces and in particular to semi-inner product spaces.

2. Main Result

Let E be a g.s.i.p. space, and suppose a is a unit vector in E

THEOREM 2.1. *If* $0\leq r\leq \mathrm{Re}[x_i, a]/[x_i,x_i]^{1/p}$, $x_i\in E$, $x_i\neq 0$, $i=1,2,\cdots,n$, *and* $1<p<\infty$, *then*

$$(*)\quad r([x_1,x_1]^{1/p}+\cdots+[x_n,x_n]^{1/p})\leq [x_1+\cdots+x_n, x_1+\cdots+x_n]^{1/p}$$

where the equality holds iff

$$(**)\quad x_1+\cdots+x_n=r([x_1,x_1]^{1/p}+\cdots+[x_n,x_n]^{1/p})a.$$

PROOF. $r([x_1,x_1]^{1/p}+\cdots+[x_n,x_n]^{1/p})\leq \mathrm{Re}[x_1,a]+\cdots+\mathrm{Re}[x_n,a]$

$$=|\mathrm{Re}[x_1,a]+\cdots+\mathrm{Re}[x_n,a]|$$
$$=|\mathrm{Re}[x_1+\cdots+x_n,a]|$$
$$\leq [x_1+\cdots+x_n,x_1+\cdots+x_n]^{1/p}[a,a]^{(p-1)/p}$$

$$= [x_1+\cdots+x_n, x_1+\cdots+x_n]^{1/p}.$$

If (**) is satisfied, then clearly equality in (*) holds. Conversely assume that the equality in (*) holds. Then it holds at every intermediate inequality in the argument just given, i.e., we have

(a) $x_1+\cdots+x_n=[x_1+\cdots+x_n, a]a$,

(b) $\mathrm{Im}[x_1+\cdots+x_n, a]=0$, and

(c) $\mathrm{Re}[x_i, a]=r[x_i, x_i]^{1/p}$, $i=1, 2, \cdots, n$.

Hence,

$$\begin{aligned}[x_1+\cdots+x_n, a] &= \mathrm{Re}[x_1+\cdots+x_n, a]\\ &= \mathrm{Re}[x_1, a]+\cdots+\mathrm{Re}[x_n, a]\\ &= r([x_1, x_1]^{1/p}+\cdots+[x_n, x_n]^{1/p}),\end{aligned}$$

which gives (**) in view of (a). This completes the proof.

COROLLARY 2.2. *Under the hypothesis of* 2.1, *we have*

(1) $r([x_1, x_1]\cdots[x_n, x_n])^{1/n}\leq n^{-p}[x_1+\cdots+x_n, x_1+\cdots+x_n]$

(2) $r(([x_1, x_1]^{k/p}+\cdots+[x_n, x_n]^{k/p})/n)^{1/k}\leq n^{-1}[x_1+\cdots+x_n, x_1+\cdots+x_n]^{1/p}$

where $k<1$ *and* $k\neq 0$. *Equality in* (1) (*or in* (2)) *holds iff* $x_1+\cdots+x_n=r([x_1, x_1]^{1/p}+\cdots+[x_n, x_n]^{1/p})a$, *and* $[x_1, x_1]=\cdots=[x_n, x_n]$.

PROOF. From ([4], page 26), we have

$$\left.\begin{array}{l}([x_1, x_1]^{1/p}\cdots[x_n, x_n]^{1/p})^{1/n}\\ (([x_1, x_1]^{k/p}+\cdots+[x_n, x_n]^{k/p})/n)^{1/k}\end{array}\right\}\leq n^{-1}([x_1, x_1]^{1/p}+\cdots+[x_n, x_n]^{1/p})$$

with equality iff $[x_1, x_1]=\cdots=[x_n, x_n]$. The result follows in view of 2.1.

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REFERENCES

[1] J. B. Diaz and F. T. Metcalf, *A Complementary triangle inequality in Hilbert and Banach spaces*, Proc. Amer. Math. Soc. 17 (1966), 88—97.

[2] G. Lumer, *Semi-inner product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29—43.

[3] B. Nath, *On a generalization of Semi-inner product spaces*, Math. J. Okayama University, 15 (1971/72), 1—6.

[4] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Camb. University Press, New York, 1959.