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QUANTUM LINEAR RECURSIVE MINIMUM MEAN-SQUARE-ERROR ESTIMATION

By Young Ho Park*

Dedicated to Prof. Chung Ki Pahk on his 60th Birthday

1. Introduction

The classical communication theories have been developed within the mathematical framework of probability theory where a communication signal is treated as a random variable. With the advent of laser much attention has been given to the formulation of optical communication theory with quantum mechanics [1]—[4]. In this paper, the quantum mechanical counterpart to the Kalman filter which is one of the most important theory in communication and control is obtained. In the next section, first, some of background is described.

1.1. Theory of measurement

The concept of quantum measurement was introduced as follows [7]. With a physically measurable quantity one may associate a Hermitian operator on a Hilbert space which is called an *observable* [7].

A possible outcome of a measurement of a physically measurable quantity is one of the eigenvalues of the Hermitian operator. However, in quantum mechanical detection and estimation problem, a more general formulation of quantum measurement is desirable to deal with the simultaneous measurement of non-commuting observables [3].

DEFINITION. The measurement of operator-valued measure $\{\tilde{V}(\Delta)\}$ is called as the *generalized measurement.*

The operator-valued measure is defined as follows.

Let $(\Omega, \mathscr{B})$ a measurable space.

Consider a map

$$\Delta \rightarrow \tilde{V}(\Delta),\ \Delta \in \mathscr{B}$$

where

1. $\tilde{V}(\Delta)$ is Hermitian nonnegative operator on Hilbert space $\mathscr{H}_s$;

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*Previously with the University of Maryland.

2. (a) $\widetilde{V}(\phi)=\Theta$ where ϕ is the empty set in Ω and Θ is the null operator in $\mathscr{H}_s$.

(b) $\sum_{i\in L}\widetilde{V}(\Delta_i)=\widetilde{V}(\sum_{i\in L}\Delta_i)$ when $\bigcap_{i\in L}\Delta_i=\phi$ and L is any index set.

(c) $\widetilde{V}(\Omega)=I$, where I is the identity operator.

DEFINITION. The measurement of projection-valued measure $\{\widetilde{E}(\Delta)\}$ is called as the *simplified measurement.*

1.2. Density operator

DEFINITION. The operator ρ which has the following properties is a *density operator*:

1. $\rho=\rho^{+}$: Hermitian
2. $\mathrm{Tr}\rho=1$: Unit trace
3. $\rho\geq 0$: Positive semidefinite.

Now let's relate quantum measurement to a classical measurement.

In order to measure $\widetilde{V}(\Delta)$, we apply some measuring device that registers a set of outcomes $\bar{v}(v_1, \cdots, v_M)$ lying in the region Δ of Ω with the conditional probability density: $\Pr\{\bar{v}\in\Delta|\bar{x}\}=\mathrm{Tr}[\rho_s(\bar{x})\widetilde{V}(\Delta)]$
where $\rho_s(\bar{x})$ is the density operator of the system and $\bar{x}$ is a vector parameter [3].

With the background presented above, we now proceed to develop the quantum linear recursive minimum mean-square-error estimation theory.

2. Quantum linear recursive minimum mean-square-error estimation

Consider the following problem: a signal at optical frequencies with an unknown random parameter x_0 is received at time $k=0$, and a measurement of the optimum observable is made. After the measurement, the cavity is cleared from the received field. A signal with unknown random parameter x_1 is received in the cavity at time $k=1$. Then, we find the optimum observable at time $k=1$ such that a linear combination of the measurements at time $k=0$ and time $k=1$ minimizes the mean-square-error. After the measurement, the cavity is cleared from the received field. Then, we proceed in similar way. More rigorous mathematical formulation is given below.

Let $\rho(x_k)$ be a density operator on $\mathscr{H}$, a Hilbert space, describing the receiver input at the time k, which depends on a real discrete random process x_k where x_k satisfies the following difference equation:

$$x_k = \phi_{k-1} x_{k-1} + w_{k-1} \tag{1}$$

where ϕ_{k-1} is a real number, $\{w_k\}$ is a zero-mean white random process with variance Q_k and $\{w_k\}$ is uncorrelated with x_0, the initial zero-mean random variable.

Using (1), the following problem may be stated. First a Hermitian operator V_0 on a Hilbert space with a density operator $\rho(x_0)$ must be chosen to minimize the mean-square-error of an estimate of x_0 which is linear in the measurement v_0 of the Hermitian operator V_0. V_1 has to be selected to minimize the mean square-error of an estimate of x_1 which is linear in the measurements v_0 and v_1. Now suppose $V_0, V_1, \cdots, V_{k-1}$ are chosen as described above. Then V_k is derived to minimize the mean-square-error of an estimate of x_k which is linear in all previous measurements $v_0, \cdots, v_{k-1}$ and the present measurement v_k of Hermitian operator V_k.

THEOREM 1. *The optimum observable V_k satisfies the following equation*

$$\eta^{(0)} V_0 + V_0 \eta^{(0)} = 2\,\delta^{(0)} \tag{2}$$

where

$$\eta^{(0)} = E_{x_0}[\rho(x_0)] \tag{3}$$

$$\delta^{(0)} = E_{x_0}[x_0 \rho(x_0)] \tag{4}$$

PROOF. See Park [5] or Personick [2].

The mean-square-error at time k can be written as follows [5], [6] :

$$\mathrm{MSE}_k = E_{x_0, \cdots, x_k}\left\{\mathrm{Tr}\,[\rho(x_0)\otimes\cdots\otimes\rho(x_k)(x_k - \sum_{i=0}^{k} c_i^{(k)} V_i^{(k)})^2]\right\} \tag{5}$$

where

$$\rho(x_0)\otimes\cdots\otimes\rho(x_k) \in \mathscr{H}^{(k)} \tag{6}$$

$$V_i^{(k)} = I\otimes\cdots\otimes I\otimes V_i \otimes I\otimes\cdots\otimes I \in \mathscr{H}^{(k)} \tag{7}$$

$$\mathscr{H}^{(k)} = \mathscr{H}_0 \otimes\cdots\otimes \mathscr{H}_k\text{: tensor product space of Hilbert spaces.} \tag{8}$$

THEOREM 2. *There exist the optimal observable V_k and the optimal processing coefficients $c_i^{(k)}$, $i=0, 1, \cdots, k-1$, if and only if there exists solution to the equations:*

$$\eta_T^{(k)} V_k + V_k \eta_T^{(k)} = 2\,[\delta_T^{(k)} - \sum_{i=0}^{k-1} c_i^{(k)} (\eta^{(k)} V_i^{(k)})_T] \tag{9}$$

where

$$\eta_T^{(k)} = \mathrm{Tr}_{0, \cdots, k-1}[\eta^{(k)}]$$

$$\delta_T^{(k)} = \mathrm{Tr}_{0, \cdots, k-1}[\delta^{(k)}]$$

$$(\eta^{(k)}V_i^{(k)})_T = \mathrm{Tr}_{0,\ \cdots,k-1}[\eta^{(k)}V_i^{(k)}]$$
$$\eta^{(k)} = E_{x_0,\ \cdots,x_k}[\rho(x_0)\otimes\cdots\otimes\rho(x_k)]$$
$$\delta^{(k)} = E_{x_0,\ \cdots,x_k}[x_k\rho(x_0)\otimes\cdots\otimes\rho(x_k)]$$
$$\sum_{i=0}^{k-1} c_i^{(k)}\mathrm{Tr}\,[\eta^{(k)}V_0^{(k)}(V_i^{(k)}-\tau_i^{(k)})] = \mathrm{Tr}\,[\delta^{(k)}V_0^{(k)} - \eta^{(k)}V_0^{(k)}\tau_k^{(k)}]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{i=0}^{k-1} c_i^{(k)}\mathrm{Tr}\,[\eta^{(k)}V_{k-1}^{(k)}(V_i^{(k)}-\tau_i^{(k)})] = \mathrm{Tr}\,[\delta^{(k)}V_{k-1}^{(k)} - \eta^{(k)}V_{k-1}^{(k)}\tau_k^{(k)}] \tag{10}$$

where

$$\tau_i^{(k)} = I\otimes\cdots\otimes I\otimes\tau_i,\quad \forall i=0,\ \cdots,\ k-1$$
$$\tau_i = 2\int e^{-\theta\eta_T(k)}(\eta^{(k)}V_i^{(k)})_T e^{-\theta\eta_T(k)}\,d\theta \ \textit{for all}\ i=0,\ \cdots,\ k-1$$
$$(\eta^{(k)}V_i^{(k)})_T = \mathrm{Tr}_{0,\ \cdots,\ k-1}\,[\eta^{(k)}V_i^{(k)}] \in \mathscr{H}_k$$
$$\tau_k^{(k)} = I\otimes\cdots\otimes I\otimes\tau_k$$
$$\tau_k = 2\int e^{-\theta\eta_T(k)}\delta_T^{(k)}e^{-\theta\eta_T(k)}\,d\theta \in \mathscr{H}_k.$$

Then the optimum V_k *is given by*

$$V_k = \tau_k - \sum_{i=0}^{k-1} c_i^{(k)}\,\tau_i. \tag{11}$$

PROOF. Defining proper inner product and using projection theorem, the result has been obtained. See Baras & Park [5], [6].

3. Example

Let's consider a real single mode of a coherent signal in thermal noise. In this case, the conditional density operator $\rho(x_i)$ of a signal received in the cavity at time i is given by

$$\rho(x_i) = \frac{1}{\pi N}\int \exp\left\{\frac{-|\alpha_i - x_i|^2}{N}\right\}|\alpha_i\rangle\langle\alpha_i|\,d^2\alpha_i \tag{12}$$

where $|\alpha_i\rangle$ is a coherent state in $\mathscr{H}_i$. Consider the dynamical model for a random process $\{x_i\}$ where the x_i's satisfy the stochastic difference equation:

$$x_i = \phi_{i-1}x_{i-1} + w_{i-1}$$

where x_0 is a Gaussian random variable with zero mean, $\{w_i\}$ is a zero-mean white Gaussian process with the variance Q_i, $\{w_i\}$ is independent of x_0 and ϕ_{i-1} is a real number.

We can prove that all V_k's are the same type of observable, i.e.,

$$V_k=e_k(a_k+a_k^+) \text{ for all } k$$

where e_k is a constant and a_k is the annilation operator on $\mathscr{H}_k$, i.e.,

$$a_k|\alpha_k\rangle=\alpha_k|\alpha_k\rangle. \tag{13}$$

DEFINITION. A random variable is defined as *the realization of the optimum observable* if the measurement of the optimum observable and the random variable have the same characteristic function.

Now, let's consider a realization of the optimum observable $a_i+a_i^+$. We can show that the conditional characteristic function of the measurement of $a_i+a_i^+$ is

$$\mathrm{Tr}\left\{\rho(x_i)\exp\left[\frac{-s(a_i+a_i^+)}{2}\right]\right\}$$
$$=e^{-sx_i}e^{s^2\left[\left(\frac{N}{2}+\frac{1}{4}\right)/2\right]}. \tag{14}$$

Observing (14), we can conclude that the measurement v_i of V_i is given by

$$v_i=e_i(x_i+n_i)$$

where n_i is a Gaussian random variable with zero-mean and variance $\left(\frac{N}{2}+\frac{1}{4}\right)$ and independent of x_i. Hence our quantum linear recursive minimum mean-square-error estimation problem becomes a classical linear recursive minimum mean-square-error estimation problem where

$$x_{k+1}=\phi_k x_k+w_k \tag{15}$$
$$y_k=x_k+n_k \tag{16}$$
$$E[w_k w_l]=Q_k\delta_{kl} \tag{17}$$
$$E[n_k n_l]=R_k\delta_{kl} \tag{18}$$
$$\delta_{kl}, \text{ Kronecker delta}$$

We will now describe recursive scheme. Let $\hat{x}_{k+1/k+1}$ be the optimum estimate of x_{k+1}, given the data $y_0, \cdots, y_{k+1}$. Then $\hat{x}_{k+1/k+1}$ can be generated recursively by the following set of equations.

$$\hat{x}_{k+1/k+1}=\phi_k\hat{x}_{k/k}+P_{k+1/k}[P_{k+1/k}+R_{k+1}]^{-1}(y_{k+1}-\phi_k\hat{x}_{k/k}) \tag{19}$$
$$P_{k+1/k+1}=P_{k+1/k}-P_{k+1/k}[P_{k+1/k}+R_{k+1}]^{-1}P_{k+1/k} \tag{20}$$
$$P_{k+1/k}=\phi_k^2 P_{k/k}+Q_k \tag{21}$$

where

$$P_{k+1/k+1}=E[(x_{k+1}-\hat{x}_{k+1/k+1})^2] \tag{22}$$

$$P_{k+1/k}=E[(x_{k+1}-\hat{x}_{k+1/k})^2] \tag{23}$$

$$R_k=E(n_k^2)=\frac{N}{2}+\frac{1}{4} \qquad \text{for all } k \tag{24}$$

$$Q_k=E(w_k^2) \tag{25}$$

4. Conclusion

Even though, only one dimensional case is presented here, it can be extended to two dimensional case. In general, it takes large amount of computation to solve k simultaneous linear equation (10) as k grows. Thus it is desirable to develop an approximate recursive scheme. One approximate scheme is given in the paper [5]. One of problems which need further research is to study whether any recursive relation exists between optimum observables V_i, $i=0, \cdots, k$. Otherwise, the measurement scheme becomes different for different observables.

Bendix Corporation
Columbia, Maryland

REFERENCES

[1] C.W. Helstrom, J.W.S. Liu, *Quantum Mechanical Communication Theory,* IEEE Proc., vol. 58 ≠10, Oct. 1970.

[2] S.D. Personick, *Efficient Analog Comm. over Quantum Channels,"* Tech Rept. 477 RLE, M.I.T., Cambridge, Mass., May 1970.

[3] C.W. Helstrom and R.S. Kennedy, *Noncommuting Observables in Quantum Detection and Estimation Theory,* IEEE Trans. vol. IT—20, No. 1, Jan. 1974.

[4] H.P. Yuen and M. Lax, *Multiple-Parameter Quantum Estimation and Measurement of Nonselfadjoint Observables,* IEEE Trans. Vol. IT-19, No. 6, Nov. 1973.

[5[Y.H. Park, Ph. D. Thesis, Aug. 1974, Univ. of Md., College Park.

[6] J.S. Baras and Y.H. Park, *Estimation of Random Signals Bases on Quantum Mechanical Measurements,* 8th Asilomor Conference on Circuits, Systems and Computers, Dec. 1974.

[7] P.A.M. Dirac, *The Principles of Quantum Mechanics,* 4th Ed., Oxford, Clarendon, 1958.

[8] R.J. Glauber, *Coherent and Incoherent States of Radiation Field,* Phys, Rev., vol 131, Sep. 1963.

[9] F. R. Gantmacher, *The Theory of Matrices*, vols. I & II, New York, John Wiley and Sons, 1970.

[10] N. Jacobson, *Lectures in Abstract Algebra*, vol. II Princeton, Van Nostrand 1953.

[11] P. R. Halmos, *Finite Dimonsional Vector Spaces*, Princeton, Van Nostrand 1970.