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THE MAXIMAL SUBGROUPS IN $L(H)$

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H.P. Decell, Jr. and C.L. Wiginton characterized the maximal subgroups of the matrix algebra of all $n\times n$ complex matrices [4]. Motivated by their work [4], we extend their result in the setting of $L(H)$, the algebra of all bounded operators on a Hilbert space H.

The next Lemma 1 was obtained in [3] (P.675 Theorem 1) for the case dim $(H)=n$, and in [1] (P.551 Proposition 2.3 (g),(k)), [2] (P.421 Theorem 1) for the general case.

LEMMA 1. *Let $T\in L(H)$. If Range (T) is closed, then the next four simultaneous equations have a unique solution X in $L(H)$, called the generalized inverse of T and denoted by T^+.*

(1) $TXT=T$ (2) $XTX=X$ (3) $(TX)^*=TX$ (4) $(XT)^*=XT$. *Morcover TT^+ is the orthogonal projection onto Range(T) and T^+T is that onto Range (T^*). Also Range$(T^+)=$Range(T^*) and these two linear subspaces arc closed. Conversely, the single condition* (1) *guarantees the closedness of Range (T).*

The proof of the next lemma is elementary and omitted.

LEMMA 2. *Let E and F be two idempotent elements of $L(H)$. If Range(E) $=$Range(F), then there is an invertible element $P\in L(H)$ such that $P^{-1}EP=F$. Furthermore P can be chosen so that $EP=F$.*

The following theorem generalizes theorem 2, P.676 and Corollary, P.677 in [4].

THEOREM 3. *Let E denote an orthogonal projection on H and let $M(E)=\{T\in L(H)$: Range$(T)=$Range$(T^*)=$Range$(E)\}$. Then G is a maximal subgroup of $L(H)$ if and only if $G=P^{-1}M(E)P$ for a suitable orthogonal projection E and invertible operator P. In this circumstance, the inverse of $T\in G$ in G is T^+.*

Proof. (Sufficiency) For each $T\in M(E)$, note that $ET=T$ and $ET^*=T^*$, so that $ET=T=TE$. Hence E serves as the identity of $M(E)$. Let $T,S\in M(E)$, then Range$(E)=$Range$(T)=$Range(TE) $=$Range$(TSS^+)\subset$Range(TS)

$\subset \text{Range}(T)=\text{Range}(E)$, by Lemma 1. It follows that $\text{Range}(TS)=\text{Range}(E)$. By the similar reason, $\text{Range}(S^*T^*)=\text{Range}(E)$, since $S^*, T^*\in M(E)$. Therefore, $TS\in M(E)$, proving $M(E)$ is closed under multiplication. To see that $T^+\in M(E)$, we first note that $(T^+)^*=(T^*)^+$, by using Lemma 1. Therefore, again with the aid of Lemma 1, $\text{Range}\,(T^+)^*=\text{Range}(T^*)^+=\text{Range}\,(T^*)^*=\text{Range}(T)=\text{Range}(E)=\text{Range}(T^*)=\text{Range}(T^+)$. Hence $T^+\in M(E)$. It follows that $M(E)$ is a group. Now let K be a subgroup of $L(H)$ such that $M(E)\subset K$. Let F be the identity element of K and E^{-1} the inverse of E in K. Then $F=EE^{-1}=E^2E^{-1}=E(EE^{-1})=EF=E$. It follows that $TT^{-1}=E$. Now $\text{Range}(E)=\text{Range}(TT^{-1})$, $\text{Range}(T)=\text{Range}(ET)\subset\text{Range}(E)$. By the fact that $M(E)\subset K^*$ and that K^* is a group, we can similarly show that $\text{Range}(T^*)=\text{Range}(E)$. Hence $T\in M(E)$. The maximality of $P^{-1}M(E)P$ follows immediately. (Necessity) Let G be a maximal subgroup of $L(H)$ with the identity F. Let E be the orthogonal projection onto $\text{Range}(F)$. By Lemma 2, there is an invertible operator $P\in L(H)$ such that $F=P^{-1}EP$. Then G and $P^{-1}M(E)P$ are two maximal subgroups of $L(H)$ with the common identity F. Therefore, $G=P^{-1}M(E)P$. Q.E.D.

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