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A BILLIARD TABLE PROBLEM IN E^n

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To understand a general case in E^n we begin with a familiar billiard table problem in E^2.

1. In E^2. We consider first a billiard table and a shot satisfying the following three conditions: (i) The table dimension is $a\times b$ where a and b are positive integers and relatively prime. (ii) One billiard ball is shot from the lower left-hand corner at a 45° angle to its sides. (iii) The ball travels indefinitely, unless it hits a corner in which case it stops.

THEOREM 1. *The ball stops after traveling a total distance of* $\sqrt{2}ab$ *while striking* $a+b-2$ *cushions, excluding the two corners of its departure and termination. If we name the four corners* 0, 1, 2, *and* 3 *as in Figure* 1 ($a=2$, $b=5$), *the ball will terminate at corner* 1 *if a is even,* 2 *if b is even, and* 3 *if both a and b are odd.*

Proof. Consider a lattice in the first quadrant tessilated by $a\times b$ rectangles as in Figure 2.

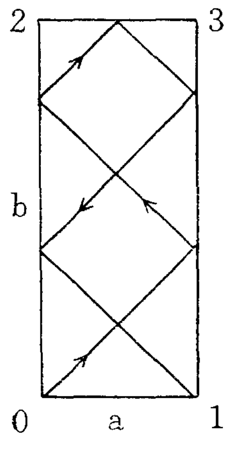

Figure 1

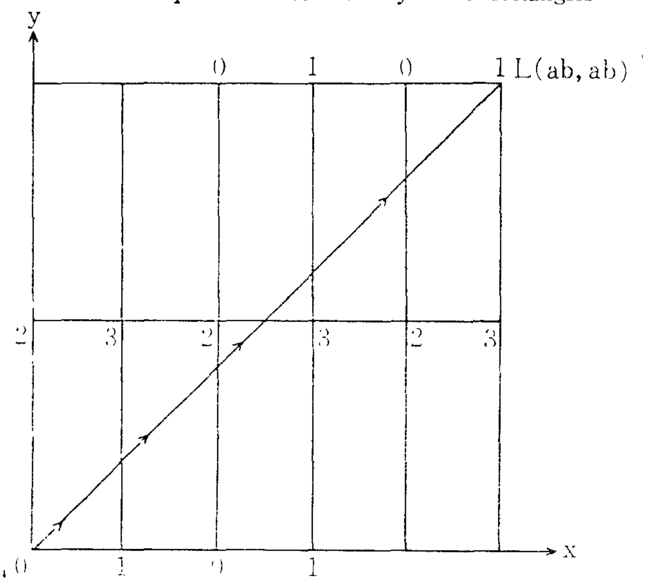

Figure 2

By reflection, there is a 1-1 correspondence between the line $x=y$ and the ball's actual path as in Figure 1, and between the intersections of the line $x=y$ with the lattice and the cushions struck by the ball. The line $x=y$ first intersects a lattice point L at (l, l), corresponding to the terminal corner, where l is the l. c. m. of a and b and hence $l=ab$, and en route intersects $l/a-1(=b-1)$ vertical and $l/b-1(=a-1)$ horizontal lattice lines. Accordingly, the ball strikes $a+b-2$ cushions before it strikes the corner.

To identify the corner which corresponds to the point L, we represent each reflection of the table by permutations $P_1=(01)\ (23)$, $P_2=(02)(13)$ and their combinations $P_1P_2=P_2P_1=(03)(12)$, I the identity. In particular, the last table position whose corners contain L is given by

$$P_1^{b-1}P_2^{a-1}$$

because the table is being reflected along the b-side $b-1$ times and the a-side $a-1$ times, and the terminal corner can be found as a number which replaces the corner 3 by P_1^{b-1}, P_2^{a-1}. Since

$$P_1^{b-1}P_2^{a-1}=\begin{cases} P_2=(02)(13) \text{ if } a \text{ is even,} \\ P_1=(01)(23) \text{ if } b \text{ is even,} \\ I \text{ if both } a \text{ and } b \text{ are odd,} \end{cases}$$

the terminal corner is 1 if a is even, 2 if b is even, or 3 if both are odd, as stated in the theorem. Note that P_1P_2 does not occur in the final position.

To extend the above theorem beyond the conditions of standardization described at the beginning, we may consider the case $(a, b)=t(a', b')$ where $t>1$ and $(a', b')=1$. Then the number of cushions and the terminal corner on the table $a\times b$ can be found from a similar table of reduced size $a'\times b'$, but the total distance traveled will be $\sqrt{2}ab/t$ $(=t(\sqrt{2}a'b'))$ on the table $a\times b$. Secondly, we change the direction of the ball. If the ball is shot toward a point $(1, r)$ for rational r, instead of $(1, 1)$ or a $45°$ from the sides of the table $a\times b$, then consider a table of size $a'\times b'$ which is similar to a table of size $a\times(b/r)$ and $(ka, kb/r)=(a', b')=1$ for some rational k. Then the number of cushions is $a'+b'-2=k\left(a+\frac{b}{r}\right)-2$ and the length of the ball's path is $(\sqrt{2}a'b'/k)(\sqrt{1+r^2}/\sqrt{2})=kab\sqrt{1+r^{-2}}$ in the original table $a\times b$.

2. In E^n. Here we will try to generalize theorem 1 in a higher dimension. In E^3, In particular, a ball's path on a billiard table can be interpreted as a path of a light beam reflected on mirrored surfaces inlaid inside a rectangular

box. Call a generalized rectangular box a hyper box in E^n and name 2^n corners by numbers $0, 1, 2, \cdots, 2^n-1$ in the following fashion: Take a corner as the origin and name it 0 (zero). Then take n edges from 0 in an arbitrary order as the $x_1, x_2, \cdots, x_n$ axes. Suppose the size of our hyper box is $a_1 \times a_2 \times \cdots \times a_n$, where a_i is a positive integer length along the x_i axis and $(a_1, a_2, \cdots, a_n) = 1$. Then the coordinate of a corner is one of the expressions

E^2	E^3	E^4	E^n
$P_1 = (01)(23)$	$(45)(67)$	$(89)(10\ 11)(12\ 13)(14\ 15)$	$\cdots(2^n-2,\ 2^n-1)$
$P_2 = (02)(13)$	$(46)(57)$	$(8\ 10)(9\ 11)(12\ 14)(13\ 15)$	$\cdots(2^n-3\ \ 2^n-1)$
$P_3 = (04)(15)\ \ (26)(37)$		$(8\ 12)(9\ 13)(10\ 14)(11\ 15)$	$\cdots(2^n-4-1\ \ 2^n-1)$
$P_4 = (08)(19)(2\ 10)(3\ 11)(4\ 12)(5\ 13)(6\ 14)(7\ 15)$			$\cdots(2^n-2^3-1\ \ 2^n-1)$
$P_n = (0\ 2^{n-1})(1\ 2^{n-1}+1)(2\ 2^{n-1}+2) \cdots\cdots(2^n-2^{n-1}-1\ \ 2^n-1)$,			

these n permutations generate a commutative group of order 2^n and the final position of the reflected box will be $\prod_{i=1}^{n} P_i^{\frac{l}{a_i}-1}$. Hence the terminal corner can be identified by the number replaced by 2^n-1 in the above final position.

Let $\prod P_i^{\frac{l}{a_i}-1} = \prod P_i^{\varepsilon_i}$ where $\varepsilon_i = 1$ or 0 according to whether $\frac{l}{a_i}-1$ is odd or even. And $\frac{l}{a_i}-1$ is odd if l/a_i is even. If we denote the exponent of 2 in the prime power representation of an integer j by $|j|_2$ (e.g. $|40|_2 = |2^3 5|_2 = 3$), l/a_i is even iff $|a_i|_2 < |l|_2$.
Hence

$$\varepsilon_i = \begin{cases} 1 & \text{if } |a_i|_2 < |l|_2, \\ 0 & \text{if } |a_i|_2 = |l|_2. \end{cases}$$

Since the permutation $P_n^{\varepsilon_n}$ causes replacement of 2^n-1 by $2^n-1-\varepsilon_n 2^{n-1}$ and $P_{n-1}^{\varepsilon_{n-1}}$ causes replacement of the latter by $2^n-1-\varepsilon_n 2^{n-1}-\varepsilon_{n-1}2^{n-2}$ again, and so on, the terminal corner will be

$$2^n-1-\sum_{i=1}^{n}\varepsilon_i 2^{i-1} = \sum 2^{i-1} - \sum \varepsilon_i 2^{i-1} = \sum_{i=1}^{n}(1-\varepsilon_i)2^{i-1}.$$

Therefore, the corner we are looking for can be expressed

$$C = \sum_{i=1}^{n}\delta_i 2^{i-1}, \quad \text{where } \delta_i = \begin{cases} 1 & \text{if } |a_i|_2 = |l|_2 \\ 0 & \text{if otherwise} \end{cases}$$

For instance, if all a_i's are odd, $\prod_i P_i^{\varepsilon_i}=\mathrm{I}$, the identity and $C=\sum_{i=1}^{n}\delta_i 2^{i-1}=\sum 2^{i-1}=2^n-1$. Among the 2^n elements of the group the permutation $P_1P_2\cdots P_n$ which would replace 2^n-1 by 0 never occurs in the final position because $\frac{l}{a_i}-1$ cannot be all odd i.e. l/a_i cannot be all even to satisfy $(a_1, a_2, \cdots, a_n)=1$.

When the ball is hit toward the point $(1, 1, \cdots, 1)$ from 0 (the distance of the two points is $\sqrt{n}$), the ball will pass the points $(1, 1, \cdots, 1)$, $(2, 2, \cdots, 2)$, $\cdots$, until it hits a side ($n-1$ dimensional box) and reflects there.

Summarizing the above argument we state the following theorem.

$$(\varepsilon_1 a_1, \varepsilon_2 a_2, \cdots, \varepsilon_n a_n) \text{ with } \varepsilon_i=0 \text{ or } 1.$$

We name this corner by the number

$$\varepsilon_1+\varepsilon_2 2+\varepsilon_3 2^2+\varepsilon_4 2^3+\cdots+\varepsilon_n 2^{n-1},$$

one of $0, 1, 2, \cdots, 2^n-1$. Figures 3 and 4 show boxes with properly named corners. In this setting the equation of the initial direction of

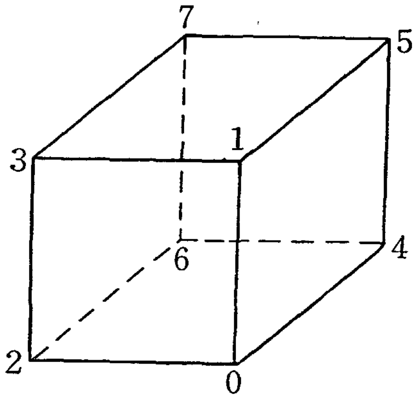

Figure 3. A box in E^3

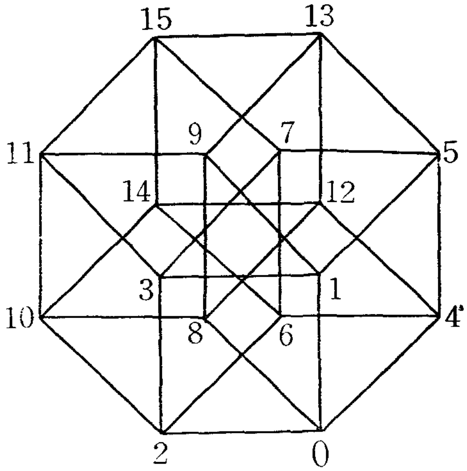

Figure 4. A box in E^4

a ball's path would be $x_1=x_2=\cdots=x_n$ and the number of cushions would be $\sum_{i=1}^{n}(l/a_i-1)$, where l is the l.c.m. of the a_1's. Since at every cushion our hyper box is reflected and its reflected position can be expressed by a combination of the following n permutations of order 2:

THEOREM 2. *In E^n, the ball stops after traveling a total distance of $\sqrt{n}l$ while striking the sides of a hyper box of size $a_1\times a_2\times\cdots\times a_n$ $\sum_{i=1}^{n}(l/a_i-1)$*

times, excluding the two corners of its departure and termination. If we name the corner $(\varepsilon_1 a_1, \varepsilon_2 a_2, \cdots, \varepsilon_n a_n)$ *with* $\varepsilon_i = 0$ *or* 1 *by* $\sum \varepsilon_i 2^{i-1}$, *then the ball will terminate at the corner* $\sum_{i=1}^{n} \delta_i 2^{i-1}$, *where* δ_i *is defined by* $\delta_i = \begin{cases} 1 & \text{if } |a_i|_2 = |l|_2, \\ 0 & \text{otherwise.} \end{cases}$

REMARKS. **1.** In case $a_1, a_2, \cdots, a_n$ are pairwise relatively prime, the distance traveled by the ball is $\sqrt{n}a_1 a_2 \cdots a_n$ and this implies that the ball has to pass through every unit hyper box $a_1 \times a_2 \times \cdots \times a_n$ once and only once. **2.** In figure 1 if we que a ball from the corner 2, it will terminate at 3. An interesting fact is that if we shoot a ball from any point on the table but not on the two known paths from a corner to another, then it will travel a loop of length exactly equal to two times that of the ball's path from corner to corner. (Shoot the ball parallel to the usual direction.)

References

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