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A GENERALIZATION OF STRATIFIABLE SPACES

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As a generalization of stratifiable spaces, Creede [4] introduces semi-stratifiable spaces. This class of spaces lies between the class of semi-metric spaces and the class of perpect spaces (closed sets are G_δ).

In what follows, all spaces are assumed to be T_1 unless otherwise mentioned and the set of positive integers is denoted by N.

A topological space X is said to be *semi-stratifiable* if, to each open set $U\subset X$, one can assign a sequence $\{U_n : n\in N\}$ of closed subsets of X such that (a) $\bigcup_{n=1}^{\infty}U_n=U$, (b) $U_n\subset V_n$ whenever $U\subset V$. (If, in addition, the space X satisfies (c) $\bigcup_{n=1}^{\infty}Int(U_n)=U$, then X is called *stratifiable* [2].)

Creede [4] has shown that a topological space X is semi-stratifiable if and only if there exists a function g from $N\times X$ into the collection of open sets of X such that

(1) for each $x\in X$, $\{g(n,x) : n\in N\}$ is a nonincreasing sequence such that $\bigcap_{n=1}^{\infty}g(n,x)=\{x\}$ and

(2) if y is a point of X and $\{x_n : n\in N\}$ is a sequence of points in X with $y\in g(n,x_n)$ for all $n\in N$, then $\{x_n : n\in N\}$ converges to y.

A topological space X is called a *semi-metric* space if there is a distance function d defined on X such that

(1) $d(x,y)=d(y,x)\geqq 0$,

(2) $d(x,y)=0$ if and only if $x=y$ and

(3) x is a limit point of a set M if and only if $d(x,M)=0$. The space X is said to be *symmetric* if d satisfies conditions (1), (2) and the following condition instead of (3).

(3′) A subset M is closed if and only if $d(x,M)>0$ for each $x\in(X-M)$.

Heath [5] showed the following

THEOREM (Heath). *A T_1-space is a semi-metric space if and only if it is a first countable semi-stratifiable space.*

LEMMA (Heath). *A necessary and sufficient conditicn that a topological space*

X be semi-metric is that there exists a function g from $N\times X$ *into the collection of open sets of X such that*

(1) *for each* $x\in X$, $\{g(n,x):n\in N\}$ *is non-increasing sequence which forms a local base for the topology at* x, *and*

(2) *if* $y\in X$ *and* $\{x_n:n\in N\}$ *is a sequence of points in X such that,* $y\in g(n, x_n)$ *for each* $n\in N$, *then* $\{x_n:n\in N\}$ *converges to* y.

Now we consider a class of spaces with the following condition (*). A topological space X has a function g from $N\times X$ into the collection of subsets of X which satisfies

(1) for each $x\in X$, $\{g(n,x):n\in N\}$ is a non-increasing sequence of subsets such that $x\in\bigcap_{n=1}^{\infty}g(n,x)$,

(2) if $y\in X$ and $\{x_n:n\in N\}$ is a sequence of points in X such that, $y\in g(n,x_n)$ for each $n\in N$, then $\{x_n:n\in N\}$ converges to y and

(3) a subset M is closed if, for any $x\in(X-M)$, there exist an open set U containing x and a positive integer n such that $U\cap g(n,x)\cap M=\phi$.

It is clear that the semi-stratifiable spaces satisfy the condition (*).

THEOREM 1. *A* T_2*-space X is semi-metric if and only if it is first countable and satisfies the condition* (*).

Proof. The necessity is clear. To prove the sufficiency, let f be a function from $N\times X$ into the collection of open sets of X such that $\{f(n,x):n\in N\}$ is non-increasing local base at x. Such a function exists since X is first countable. Let g be the function in condition (*), and $h(n,x)=f(n,x)\cap g(n,x)$. Then, by the previous Lemma, it is sufficient to show that $x\in Int[h(n,x)]$, for any $n\in N$ and $x\in X$. Suppose that there exist $k\in N$ and $x\in X$ such that $f(n,x)-h(n,x)\neq\phi$ for each $n\in N$. Then for each $n\in N$, we can choose $x_n\in[f(n,x)-h(n,x)]$. Since $\{x_n:n\in N\}$ converges to x and X is T_2-space, the set $F=\{x_n:n\in N\}\cup\{x\}$ is closed. Hence for any $z\in(X-F)$, there is $m\in N$ such that $h(m,z)\cap F=\phi$. Since $h(k,x)\cap(F-\{x\})=\phi$, for any $z\in[X-(F-\{x\})]$, there is a positive integer n such that $h(n,z)\cap(F-\{x\})=\phi$. This means that $F-\{x\}$ is closed. But $x\in cl(F-\{x\})$. This is a contradiction.

THEOREM 2. *A topological space X is symmetric if and only if there is a function g from* $N\times X$ *into the collection of subsets of X which satisfies*

(1) $\{g(n,x):n\in N\}$ *is non-increasing local net at* x (*i.e.* $x\in\bigcap_{n=1}^{\infty}g(n,x)$

and for any neighborhood U of x, there is $n\in N$ such that $g(n,x)\subset U$),

(2) *if y is a point of X and $\{x_n : n\in N\}$ is a sequence of points in X with $y\in g(n,x_n)$ for all $n\in N$, then $\{x_n : n\in N\}$ converges to y and*

(3) *a subset M is closed if, for any $x\in(X-M)$, there exists $n\in N$ such that $g(n,x)\cap M=\phi$.*

Proof. Necessity. Let d be the distance function and $g(n,x)=\{y\in X : d(x,y)<\frac{1}{n}\}$. Then clearly $\{g(n,x) : n\in N\}$ is a non-increasing local net at x. Hence, for any $n\in N$, $y\in g(n,x_n)$ means $\{x_n : n\in N\}$ converges to y since $x\in g(n,y)$ if and only if $y\in g(n,x)$. The remaining part is clear.

Sufficiency. Let $m(x,y)=min\{j\in N : y\notin g(j,x)\}$, and define a distance function d for X as follows: if $x\in X$, $d(x,x)=0$; if x and y are two points of X $d(x,y)=min\{1/m(x,y),\ 1/m(y,x)\}$ $(=1/min\{j\in N : y\notin g(j,x)$ and $x\notin g(j,y)\})$. Clearly $d(x,y)=d(y,x)$, and $d(x,y)\geqq 0$. It remains to prove that a subset M is closed if and only if $d(x,M)>0$ for any $x\in(X-M)$. If there is $x\in(X-M)$ such that $d(x,M)=0$, then there exists a sequence $\{x_n : n\in N\}$ in M such that $d(x,x_n)<\frac{1}{n}$, for each $n\in N$. Hence $min\{j\in N : x_n\notin g(j,x)$ and $x\notin g(j,x_n)\}>n$, for each $n\in N$. This means that $x_n\in g(n,x)$ or $x\in g(n,x_n)$, for each $n\in N$. Therefore we can show that $x\in cl\{x_n : n\in N\}$. Since $cl\{x_n : n\in N\}\subset cl(M)$ and $x\notin M$, M is not closed.

Conversely, assume that M is not closed, then there exists a point $x\in(X-M)$ such that $g(n,x)\cap M\neq\phi$ for each $n\in N$. This means that there is a sequence $\{x_n : n\in N\}$ in M such that $d(x,x_n)<\frac{1}{n}$ for each $n\in N$. Hence $d(x,M)=0$.

From the above Theorems, it is clear that any symmetric space satisfies the condition (*), and hence we have

COROLLARY (Burke). *A T_2-space X is semi-metric if and only if it is first countable and symmetrizable.*

Thus, Theorem 1 is a formal generalization of Theorem (Heath) and the above Corollary. In fact a space with the condition (*) is distinguished from the semi-stratifiable spaces as shown in the following

EXAPLE 3. In [1] Bonnett gives an example of a symmetrizable space which is not perpect. This space is not semi-stratifiable, but it satisfies the condition (*).

References

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