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AN APPLICATION OF PROPER MAPS ON TOPOLOGICAL GROUPS

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We outline in this paper an application of proper maps on some topological groups. A simple application of the maps is as follows: if C is a topological group and H is a compact subgroup of G, then the natural map ϕ of G onto G/H is a closed map; therefore, ϕ is a proper map, and hence, G is compact if G/H is.

DEFINITION. ([1], p. 97) Let f be a map of a topological space Z into a topological space Y. f is said to be *proper* if f is continuous and $f\times i_z : X\times Z\to Y\times Z$ is closed for all topological spaces Z, where i_z is the identity map on Z.

LEMMA 1. ([1], p. 101) *Let $f : X\to Y$ be continuous. Then the following two statements are equivalent*:

a) f is proper

b) f is closed and $f^{-1}(y)$ is compact for each y in Y.

LEMMA 2. ([1], p. 104) *Let $f : X\to Y$ be a proper mapping, and let K be a compact subset of Y. Then $f^{-1}(K)$ is compact.*

The following is an application of the above lemmas to prove a theorem in [2] ([2], (5. 23 p. 38) which avoids the use of nets.

THEOREM. *Let F be a topological group, H a subgroup of G, and ϕ the natural mapping of G onto the left-coset space G/H. Suppose that U is a symmetric neighborhood of the identity e such that the following hold*:

i) $(\bar{U}^3)^- \cap H$ *is compact*

ii) $\{xH: x\in X\}$ *is a closed compact subset of G/H and*

iii) $\{xH: x\in X\}\subset\{uH: u\in U\}$.

Then $\bar{U}\cap XH$ is closed and compact in G.

Proof. $XH=\phi^{-1}(\{xH: x\in X\})$ is closed; hence, $\bar{U}\cap XH$ is closed. Consider the restriction $\psi=\phi|\bar{U}\cap XH$. By iii) $\psi(\bar{U}\cap XH)=\phi(X)$.
Thus, $\psi : \bar{U}\cap XH\to\phi(X)$ is a continuous surjection. Since the range is com-

pact, Lemma 2 will imply the domain is compact, as soon as we show that ϕ is proper, i.e., by Lemma 1, that ϕ is closed and $\phi^{-1}(\tilde{y})$ is compact for each $\tilde{y}$ in $\phi(X)$.

I) $\phi^{-1}(\tilde{y})$ is compact for each $\tilde{y}$ in $\phi(X)$:

Let $\tilde{y}\in\phi(X)$. Then $\tilde{y}=\{yH\}$ for some y in $\bar{U}$.

$$\phi^{-1}(\tilde{y})=(\bar{U}\cap XH)\cap\phi^{-1}(\tilde{y})=(\bar{U}\cap XH)\cap(y\cdot H)=\bar{U}\cap(y\cdot H)=$$

$$=y\cdot[(y^{-1}\bar{U})\cap H]=y\cdot[(y^{-1}\bar{U})\cap(\bar{U}^{3-}\cap H)]$$

Thus, $\phi^{-1}(\tilde{y})$ is a translate of the intersection of a closed set and a compact set; hence, is compact.

II) ϕ is a closed mapping:

Let C be any closed set in $(\bar{U}\cap XH)$.

Let $\tilde{x}\in\phi(x)\backslash\phi(C)$. Then $\tilde{x}=\{x_0H\}$ for some x_0 in $\bar{U}$, and $(x_0\cdot H)\cap(C\cdot H)=\phi^{-1}(\{\tilde{x}\}\cap\phi(C))=\phi$.

Let $A=(\bar{U}^3)^-\cdot x_0\cdot(A\cap H)$ is a translate of a compact set; hence, is compact, and is disjoint from the closed set C. Since G is a topological group, there exists an open symmetric neighborhood V of the identity e such that $V\subset U$ and

$$* \ [V\cdot x_0\cdot(A\cap H)]\cap C=\phi \quad \text{(see Theorem (4.10) of [2])}$$

Then $\phi(V\cdot x_0)=\{vx_0H:y\in V\}$ is an open neighborhood of $\tilde{x}$ in G/H, and $\phi(V\cdot x_0)\cap\phi(X)$ is a relatively open neighborhood of $\tilde{x}$ in the subspace $\phi(X)$. Let us now show that $[\phi(V\cdot x_0)\cap\phi(X)]\cap\phi(C)=\phi(V\cdot x_0)\cap\phi(C)=\phi$:

If not, then there exists a $v\in V$, $h\in H$ and $c\in C$ such that $vx_0h=c$. But then $h=x_0^{-1}v^{-1}c\in U^{-1}\cdot V^{-1}\cdot\bar{U}\subset\bar{U}^3\subset A$ and so $h\in A\cap H$, and $c\in[V\cdot x_0\cdot(A\cap H)]\cap C\neq\phi$. This would contradict (*).

Thus, the complement of $\phi(C)$ is open in $\phi(X)$, $\phi(C)$ is closed in $\phi(X)$, and ϕ is a closed mapping.

References

[1] Bourbaki, *General Topology*.

[2] Hewitt and Ross, *Abstract Harmonic Analysis*.

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