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A GENERALIZATION OF TYCHONOFF PRODUCT THEOREM

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The well-known Tychonoff Product Theorem says that the product of compact spaces is compact. In our best knowledge this theorem depends on the product topology (so called the Tychonoff topology) and this topology is particular to the topologies induced from the right or the initial topologies. Motivated by this fact we generalize the theorem.

Given a set X and a family of topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$ and functions $f_\alpha : X\rightarrow X_\alpha$, the topology induced on X from the right by $\{f_\alpha\}$ is the smallest or coarsest topology such that f_α is continuous.

The following generalization of [3, Theorem 12.9] is necessary to prove our main theorem.

LEMMA. *Let X have the topology induced from the right by a family of functions $\{f_\alpha : X\rightarrow X_\alpha | \alpha\in\Lambda\}$. Then a filter $\mathfrak{F}$ in X converges to $X_0\in X$ iff $f_\alpha(\mathfrak{F})$, the filter generated by all $f_\alpha(F)$, $F\in\mathfrak{F}$, converges to $f_\alpha(X_0)$ in X_α for each α.*

Proof. If $\mathfrak{F}\rightarrow X_0$ in X, then $f_\alpha(\mathfrak{F})\rightarrow f_\alpha(x_0)$ in X_α, because f_α is continuous.

Conversely, suppose $f_\alpha(\mathfrak{F})\rightarrow f_\alpha(x_0)$, for each α. Let W be any basic neighborhood of x_0 in X. Then W is of the form $f_{\alpha_1}^{-1}(U_1)\cap\cdots\cap f_{\alpha_n}^{-1}(U_n)$, where U_i is a neighborhood of $f_{\alpha_i}(x_0)$ in x_{α_i}, for $i=1,\cdots,n$. So $U_i\in f_{\alpha_i}(\mathfrak{F})$ for each i, and hence $f_{\alpha_i}(F_i)\subset U_i$ for some $F_i\in\mathfrak{F}$. Let us take $F=F_1\cap\cdots\cap F_n\in\mathfrak{F}$ and $F\subset\bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_i)=W$, so $W\in\mathfrak{F}$. Thus $\mathfrak{F}\rightarrow x_0$.

THEOREM. *Given a set X and a family of topological spaces $\{X_\alpha\}_{\alpha\in\Lambda}$ and functions $f_\alpha : X\rightarrow X_\alpha$, let X have the topology induced from the right by $\{f_\alpha\}$ such that*

(i) *$f_\alpha(X)$ is a closed subset of X_α for each $\alpha\in\Lambda$.*

(ii) *For every choice of $x_\alpha\in f_\alpha(X)\subset X_\alpha$, $\bigcap_\alpha f_\alpha^{-1}(x_\alpha)\neq\phi$*

Then, if each X_α is compact, it follows that X is compact.

Proof. It is sufficient to show that any maximal filter in X is convergent. Let $\mathcal{F}$ be a maximal filter in X. Then $f_\alpha(\mathcal{F})$ is a maximal filter in x_α for each $\alpha\in\Lambda$. Since X_α is compact, $f_\alpha(\mathcal{F})$ converges to some point $x_\alpha\in X_\alpha$ and x_α must belong to $f_\alpha(X)=\overline{f_\alpha(X)}$. For, if $x_\alpha\notin\overline{f_\alpha(X)}$, then for every neighborhood U of x_α there exists an $F\in\mathcal{F}$ such that $f_\alpha(F)\subset U$, this is impossible. Then by (ii) there exists $x\in X$ such that $f_\alpha(x)=x_\alpha$ for each $\alpha\in\Lambda$. Therefore, by Lemma, we have $\mathcal{F}\to x$.

Since the subspace topology is the one induced from the right by the inclusion, we obtain

COROLLARY 1. *Every closed subspace of a compact space is compact.*

Now the following theorem is clear.

COROLLARY 2. *Let X be a set, X_α, $\alpha\in\Lambda$, topological spaces and $f_\alpha: X\to X_\alpha$ surjections for each α such that $\bigcap_\alpha f_\alpha^{-1}(x_\alpha)\neq\phi$ for every choice of $x_\alpha\in X_\alpha$. If X has the topology induced from the right by $\{f_\alpha\}$, then X is compact iff each X_α is compact.*

COROLLARY 3. *The Tychonoff Product Theorem and its converse.*

REMARK 1. We can prove Theorem 1 just following one of the standard proofs of the Tychonoff Product Theorem, for example [1], with slight modification. The proof requires one of the equivalent forms of the Axiom of Choice.

REMARK 2. We can also prove Theorem 1 using Corollary 1 and the Tychonoff Theorem: Under the same notation in Theorem 1, $\{f_\alpha\}$ determines a function $f: X\to Y=\Pi_{\alpha\in\Lambda}X_\alpha$ such that $p_\alpha f(x)=f_\alpha(x)$ for each $x\in X$, where p_α is the α-th projection. Then clearly the topology of X is induced from the right by f and $f(X)=\Pi_\alpha f_\alpha(X)$ is compact. We can prove easily that if X has the topology induced from the right by a surjection $f: X\to X'$ and X' is compact, then X is compact.

REMARK 3. It is well-known that the Tychonoff Theorem implies the Axiom of Choice [2]. Therefore, from the above remarks, we know that the followings are equivalent.

(1) The Axiom of Choice.

(2) Theorem

(3) Corollary 2.

(4) The Tychonoff Product Theorem.

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References

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[2] Kelley, J. L., *The Tychonoff Product Theorem Implies the Axiom of Choice*, Fund. Math. **37**, (1950) 75-76.

[3] Willard, S., *General Topology*, Addison-Wesley, Reading (1970).

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