A GENERALIZATION OF Tychonoff PRODUCT THEOREM

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The well-known Tychonoff Product Theorem says that the product of compact spaces is compact. In our best knowledge this theorem depends on the product topology (so called the Tychonoff topology) and this topology is particular to the topologies induced from the right or the initial topologies. Motivated by this fact we generalize the theorem.

Given a set $X$ and a family of topological spaces $\{X_a\}_{a \in A}$ and functions $f_a : X \rightarrow X_a$, the topology induced on $X$ from the right by $\{f_a\}$ is the smallest or coarsest topology such that $f_a$ is continuous.

The following generalization of [3, Theorem 12.9] is necessary to prove our main theorem.

**Lemma.** Let $X$ have the topology induced from the right by a family of functions $\{f_a : X \rightarrow X_a | a \in A\}$. Then a filter $\mathcal{F}$ in $X$ converges to $x_0 \in X$ iff $f_a(\mathcal{F})$, the filter generated by all $f_a(F)$, $F \in \mathcal{F}$, converges to $f_a(x_0)$ in $X_a$ for each $a$.

**Proof.** If $\mathcal{F} \rightarrow x_0$ in $X$, then $f_a(\mathcal{F}) \rightarrow f_a(x_0)$ in $X_a$, because $f_a$ is continuous.

Conversely, suppose $f_a(\mathcal{F}) \rightarrow f_a(x_0)$, for each $a$. Let $W$ be any basic neighborhood of $x_0$ in $X$. Then $W$ is of the form $f_n^{-1}(U_i) \cap \cdots \cap f_1^{-1}(U_n)$, where $U_i$ is a neighborhood of $f_n(x_0)$ in $x_n$, for $i = 1, \cdots, n$. So $U_i \in f_n(\mathcal{F})$ for each $i$, and hence $f_n(F_i) \subset U_i$ for some $F_i \in \mathcal{F}$. Let us take $F = F_1 \cap \cdots \cap F_n \in \mathcal{F}$ and $F \subset \bigcap_i U_i$, $f_n^{-1}(U_i) = W$, so $W \in \mathcal{F}$. Thus $\mathcal{F} \rightarrow x_0$.

**Theorem.** Given a set $X$ and a family of topological spaces $\{X_a\}_{a \in A}$ and functions $f_a : X \rightarrow X_a$, let $X$ have the topology induced from the right by $\{f_a\}$ such that

(i) $f_a(X)$ is a closed subset of $X_a$ for each $a \in A$.

(ii) For every choice of $x_0 \in f_a(X) \subset X_a$, $\bigcap f_a^{-1}(x_0) \neq \emptyset$.

Then, if each $X_a$ is compact, it follows that $X$ is compact.
Proof. It is sufficient to show that any maximal filter in $X$ is convergent. Let $\mathcal{F}$ be a maximal filter in $X$. Then $f_\alpha(\mathcal{F})$ is a maximal filter in $x_\alpha$ for each $\alpha \in \Lambda$. Since $X_\alpha$ is compact, $f_\alpha(\mathcal{F})$ converges to some point $x_\alpha \in X_\alpha$ and $x_\alpha$ must belong to $f_\alpha(X) = \overline{f_\alpha(X)}$. For, if $x_\alpha \notin \overline{f_\alpha(X)}$, then for every neighborhood $U$ of $x_\alpha$ there exists an $F \in \mathcal{F}$ such that $f_\alpha(F) \subset U$, this is impossible. Then by (ii) there exists $x \in X$ such that $f_\alpha(x) = x_\alpha$ for each $\alpha \in \Lambda$. Therefore, by Lemma, we have $\mathcal{F} \rightarrow x$.

Since the subspace topology is the one induced from the right by the inclusion, we obtain

**Corollary 1.** Every closed subspace of a compact space is compact.

Now the following theorem is clear.

**Corollary 2.** Let $X$ be a set, $X_\alpha$, $\alpha \in \Lambda$, topological spaces and $f_\alpha : X \rightarrow X_\alpha$ surjections for each $\alpha$ such that $\bigcap \{ f_\alpha^{-1}(x_\alpha) \neq \emptyset \}$ for every choice of $x_\alpha \in X_\alpha$. If $X$ has the topology induced from the right by $\{ f_\alpha \}$, then $X$ is compact iff each $X_\alpha$ is compact.

**Corollary 3.** The Tychonoff Product Theorem and its converse.

**Remark 1.** We can prove Theorem 1 just following one of the standard proofs of the Tychonoff Product Theorem, for example [1], with slight modification. The proof requires one of the equivalent forms of the Axiom of Choice.

**Remark 2.** We can also prove Theorem 1 using Corollary 1 and the Tychonoff Theorem: Under the same notation in Theorem 1, $\{ f_\alpha \}$ determines a function $f : X \rightarrow \bigcap \{ f_\alpha(X) \}$ such that $p_\alpha f(x) = f_\alpha(x)$ for each $x \in X$, where $p_\alpha$ is the $\alpha$-th projection. Then clearly the topology of $X$ is induced from the right by $f$ and $f(X) = \bigcap \{ f_\alpha(X) \}$ is compact. We can prove easily that if $X$ has the topology induced from the right by a surjection $f : X \rightarrow X'$ and $X'$ is compact, then $X$ is compact.

**Remark 3.** It is well-known that the Tychonoff Theorem implies the Axiom of Choice [2]. Therefore, from the above remarks, we know that the followings are equivalent.

(1) The Axiom of Choice.
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(2) Theorem

(3) Corollary 2.

(4) The Tychonoff Product Theorem.

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References


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