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A NOTE ON $K_0(I)$

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Let I be a two sided ideal in an associative ring R with unit. Then there is an exact sequence of abelian groups.

(1) $$K_1(R)\to K_1(R/I)\to K_0(I)\to K_0(R)\to K_0(R/I)$$

of length five.

For the definitions of the group $K_0(I)$, the functors K_0, K_1 and for the proof see Milnor [2].

In the case that R is the ring of integers in a finite extension field of rational numbers the following theorem is to be proved, in this note, and for certain ideals I in the rings $\boldsymbol{Z}$, $\boldsymbol{Z}(\sqrt{-5})$, $K_0(I)$ are to be computed.

As a matter of notation, $U(A)$ denotes the group of units for an associative ring A. For a group homomorphism h, $\ker(f)$ $(\mathrm{coker}(f))$ denotes the kernel (the cokernel) of f.

THEOREM. *If I is an ideal of the ring R of integers in a finite extension field of rational numbers. Then we are given a short exact sequence of abelian groups.*

(2) $$0\to \mathrm{coker}(U(R)\to U(R/I))\to K_0(I)\to \mathrm{cl}(R)\to 0.$$

In particular, we have

(3) $$K_0(I)\cong \mathrm{coker}\ (U(R)\to U(R/I))\oplus \mathrm{cl}(R)$$

if the orders of the abelian groups, $\mathrm{coker}(U(R)\to U(R/I))$, $\mathrm{cl}(R)$ *are relatively prime.*

Proof. Since R is the ring of integers in a finite extension field of rational numbers the ideal class group $\mathrm{cl}(R)$ of R is finite and $K_1(R)\cong U(R)$, moreover we have $K_0(R)\cong \boldsymbol{Z}\oplus \mathrm{cl}(R)$ [2].

On the other hand, if I is a non-zero ideal in the ring R, then R/I is finite [2], and so $K_0(R/I)$ is finitely generated free abelian group, since R/I is a Artin ring [4]. Furthermore, we have $K_1(R/I)\cong U(R/I)$, since R/I is semilocal Milnor [2].

Now note the following exact sequence of abelian groups, which is induced from the exact sequence (1):

(4) $$U(R)\to U(R/I)\to K_0(I)\to K_0(R)\to K_0(R/I)$$

where $U(R)\to U(R/I)$ is the canonical group homomorphism.

$K_0(R/I)$ is torsion-free, $K_0(R)\cong \boldsymbol{Z}\oplus \mathrm{cl}(R)$ and the homomorphism $K_0(R)\to K_0(R/I)$ is not the zero homomorphism. Therefore $\ker(K_0(R)\to K_0(R/I))$ is isomorphic to the ideal class group $\mathrm{cl}(R)$.

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Thus, the short exact sequence (2) follows now immediately from the exact sequence (4). This completes the proof.

The following simple lemma is needed to compute $K_0(I)$.

LEMMA. *Let $\boldsymbol{Z}$ be the ring of integers. Let p be a prime number $\geqq 3$. Then the group $U(\boldsymbol{Z}/p^n\boldsymbol{Z})$ of units is a cyclic group of order $p^{n-1}(p-1)$ for $n\geqq 1$. For $p=2$, we have that $U(\boldsymbol{Z}/2^n\boldsymbol{Z})$ is an abelian group of type $(2^{n-2}, 2)$ for $n\geqq 3$ Speiser* [3].

Let I be the ideal $2^n\boldsymbol{Z}$ in the ring $\boldsymbol{Z}$, where $n\geqq 3$. Then $U(\boldsymbol{Z}/I)$ is an abelian group of type $(2^{n-2}, 2)$ by the Lemma. On the other hand $U(\boldsymbol{Z})=\{1, -1\}$, $\mathrm{cl}(\boldsymbol{Z})=0$ Hence $K_0(2^n\boldsymbol{Z})$ is a cyclic group of order 2^{n-2} by the exact sequence (2) The following two results are obvious, $K_0(2\boldsymbol{Z})=0$, $K_0(2^2\boldsymbol{Z})=0$

Let R be the ring $\boldsymbol{Z}(\sqrt{-5})$. Let I be the principal ideal $(2-\sqrt{-5})$ in the ring $\boldsymbol{Z}(\sqrt{-5})$. Then we have $U(R)=\{1, -1\}$, $\mathrm{cl}(R)$ is a group of order 2, and R/I is a cyclic group of order 3^2, and so $\mathrm{coker}(U(R)\to U(R/I))$ is a cyclic group of order 3. Hence

$$K_0((2-\sqrt{-5}))\cong \boldsymbol{Z}/3\boldsymbol{Z}\oplus\boldsymbol{Z}/2\boldsymbol{Z}$$

where $\boldsymbol{Z}/3\boldsymbol{Z}$, $\boldsymbol{Z}/2\boldsymbol{Z}$ denote the additive cyclic groups.

References

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