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ON LANDSBERG SPACES OF SCALAR CURVATURE

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The concept of Finsler space of scalar curvature was introduced and studied by L. Berwald in one of his posthumous papers ([3][1]). He derived the characteristic form of the curvature tensor of such a space. On the other hand he gave the concept of Landsberg space in his earlier paper ([2]).

In the present paper, combining these two concepts, we shall show the following theorem.

THEOREM 1. *Let $F^n(n\geqq 3)$ be a Landsberg space of scalar curvature R. Then F^n is a Riemannian space of constant curvature, provided $R\neq 0$.*

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The proof of the theorem depends on the following lemma, theorem and a theorem due to M. Matsumoto [5]. We are concerned with the Berwald connection and its Bianchi identities.

Let $H_j{}^i{}_{kl}$, $G_j{}^i{}_{kl}$ and $R_j{}^i{}_k$ be the h-curvature tensor, hv-curvature tensor and $(v)h$-torsion tensor respectively. (Throughout the paper we shall use the notations of [7] and the terminologies of [4].) The h- and v- covariant differentiations $X^i{}_{j;k}$, $X^i{}_{j\|k}$ of, for example, (1, 1)-tensor field $X^i{}_j(x,y)$ $(x=(x^i),\ y=(y^i),\ y^i=\dot{x}^i)$ are defined by

$$X^i{}_{j;k}=\partial X^i{}_j/\partial x^k-(\partial X^i{}_j/\partial y^r)G^r{}_k+X^r{}_jG_r{}^i{}_k-X^i{}_rG_j{}^r{}_k,$$

$$X^i{}_{j\|k}=\partial X^i{}_j/\partial y^k,\quad G^i{}_j=G_0{}^i{}_j,$$

where $G_j{}^i{}_k$ are the Berwald connection coefficients and the suffix $_0$ means the contraction by the element of support y^i. The non-trivial Bianchi identities of the Berwald connection ([4, §20]) are as follows:

(1) $$H_j{}^i{}_{kl}+H_k{}^i{}_{lj}+H_l{}^i{}_{jk}=0,$$

(2) $$H_j{}^i{}_{kl;m}+H_j{}^i{}_{lm;k}+H_j{}^i{}_{mk;l}+G_j{}^i{}_{kr}R_l{}^r{}_m+G_j{}^i{}_{lr}R_m{}^r{}_k+G_j{}^i{}_{mr}R_k{}^r{}_l=0,$$

(3) $$R_j{}^i{}_{k\|l}=H_l{}^i{}_{jk},$$

(4) $$G_l{}^i{}_{km;j}-G_l{}^i{}_{jm;k}+H_l{}^i{}_{jk\|m}=0,$$

(5) $$G_j{}^i{}_{kl}=G_k{}^i{}_{jl}=G_l{}^i{}_{kj},$$

(6) $$G_j{}^i{}_{kl\|m}=G_j{}^i{}_{km\|l}.$$

DEFINITION. A Finsler space is called an *affinely connected space*, if the coefficients $G_j{}^i{}_k$

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are functions of position only, that is, the hv-curvature tensor $G_j{}^i{}_{kl}=0$. In this paper, such a space will be called a *Berwald space*, following V. Wagner ([8]).

It is well-known that the condition $G_j{}^i{}_{kl}=0$ is equivalent to $C_{ijk|l}=0$, where C_{ijk} is the Cartan torsion tensor of the Cartan connection $C\Gamma$, and the symbol $(_|)$ denotes the h-covariant differentiation with respect to $C\Gamma$.

DEFINITION. A Finsler space is called a *Landsberg space*, if the condition $y_rG_j{}^r{}_{kl}=0$ holds good.

The condition $y_rG_j{}^r{}_{kl}=0$ is equivalent to $C_{jkl|o}=P_{jkl}=0$ or $P_{ijkl}=0$, where P_{jkl} and P_{ijkl} are the $(v)hv$-torsion tensor and hv-curvature tensor of $C\Gamma$ respectively.

DEFINITION. A Finsler space with the fundamental function $L(x, y)$ is said to be *of scalar curvature* R, if the tensor $R_{iok}=g_{ir}R_o{}^r{}_k$ is written in the form $R_{iok}=RL^2h_{ik}$, where we put $h_{ij}=g_{ij}-L^{-2}y_iy_j$. If the scalar curvature R is constant, then the space is said to be *of constant curvature*.

For such a space, the $(v)h$-torsion tensor $R_j{}^i{}_k$ and h-curvature tensor $H_l{}^i{}_{jk}$ are given by

$$3R_j{}^i{}_k=h^i{}_kB_{oj}-h^i{}_jB_{ok}, \tag{7}$$

$$\begin{aligned} 3H_l{}^i{}_{jk}&=3R_j{}^i{}_{k\|l} \\ &=3R(h_{jl}h^i{}_k-h_{kl}h^i{}_j)-h^i{}_l(B_{jk}-B_{kj}) \\ &\quad+L^{-2}y^i(h_{jl}B_{ok}-h_{kl}B_{oj})-h^i{}_jB_{lk}+h^i{}_kB_{lj}, \end{aligned} \tag{8}$$

where we put $h^i{}_j=g^{ir}h_{jr}$, and $B_{ij}=L^2R_{\|i\|j}+3R_{\|i}y_j+2R_{\|j}y_i+3RL^{-2}y_iy_j$.

DEFINITION. A Finsler space F^n is called *C-reducible*, if the tensor C_{ijk} is of the form

$$C_{ijk}=(1/n+1)(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j),$$

where C_i is the contracted torsion tensor $C_i{}^j{}_j$.

The first step to prove Theorem 1 is to show the following:

LEMMA. *If F^n is a Landsberg space of non-zero scalar curvature, then F^n is a C-reducible Finsler space.*

Proof: We shall be concerned with one of the Bianchi identities (4). Contracting (4) by y^k, we obtain

$$G_l{}^i{}_{jm\,;\,o}-H_l{}^i{}_{jk\|m}y^k=0. \tag{9}$$

Let us substitute from (8) into (9). After long computation, the equation (9) is rewritten in the following form:

$$\begin{aligned} &3G_{lijm\,;\,o}+h_{ij}K_{lm}+h_{il}K_{mj}+h_{im}K_{jl} \\ &\qquad-2y_i(h_{jl}R_{\|m}+h_{lm}R_{\|j}+h_{mj}R_{\|l}+3RC_{jlm})=0, \end{aligned} \tag{10}$$

where we put $K_{ij}=L^2R_{\|i\|j}+R_{\|i}y_j+R_{\|j}y_i$. Then contraction of (10) by y^i gives immediately

(11) $$3(y^i G_{lijm})_{;o} - 2L^2(h_{jl}R_{\|m} + h_{lm}R_{\|j} + h_{mj}R_{\|l} + 3RC_{jlm}) = 0.$$

Since $y^i G_{lijm}=0$ for the Landsberg space F^n, the equation (11) leads us to

(12) $$C_{jlm} = -(1/3R)(h_{jl}R_{\|m} + h_{lm}R_{\|j} + h_{mj}R_{\|l}),$$

provided that the scalar curvature R don't vanish. Contraction of this by g^{lm} yields easily

(13) $$R_{\|j} = -(3/n+1)RC_j.$$

It then follows from (12) and (13) that F^n is C-reducible. Consequently the proof of Lemma is complete.

The equation (13) is a generalization of $L(\partial R/\partial y^i) = 3v_2 RC^{-1}(\partial \log g/\partial y^i)$ in [6, p. 248].

THEOREM 2. *Let F^n ($n \geqq 3$) be a Berwald space of scalar curvature R. Then F^n is a Riemannian space of constant curvature or a locally Minkowski space, according to $R \neq 0$ or $R=0$.*

Proof: According to Lemma, a Berwald space F^n of non-zero scalar curvature R is C-reducible, and the equations $G_{lijm}=0$ and (12) hold good. Therefore (10) is reduced to

$$h_{ij}K_{lm} + h_{il}K_{mj} + h_{im}K_{jl} = 0.$$

Contraction of this by g^{im} leads us to $K_{jl}=0$, that is,

(14) $$L^2 R_{\|j\|l} + R_{\|j}y_l + R_{\|l}y_j = 0.$$

Substituting from (13) into (14), we obtain

(15) $$L^2(C_{j\|l} - (3/n+1)C_jC_l) + C_jy_l + C_ly_j = 0.$$

While it is known [5] that there exists a scalar α in a C-reducible Finsler space such that

(16) $$L^2(C_{j\|l} - (2/n+1)C_jC_l) + C_jy_l + C_ly_j = \alpha h_{jl}.$$

Subtracting (15) from (16), we obtain

$$(L^2/n+1)C_jC_l = \alpha h_{jl},$$

which implies $\alpha=0$, because the rank of the matrix (h_{ij}) is equal to $n-1 \geqq 2$. Therefore we have $C_j=0$, and hence $C_{ijk}=0$ from the C-reducibility, so that F^n is Riemannian. Then the equation (13) gives $R_{\|j}=0$. By virtue of Theorem in § 16 of the paper [3], it is concluded that the scalar curvature R is constant, provided dimension $n \geqq 3$. Consequently the proof of Theorem 2 is complete.

Now we shall show Theorem 1. By virtue of Lemma and Theorem 1 of the paper [5], a Landsberg space F^n ($n \geqq 3$) of non-zero scalar curvature turns out to be a Berwald space. Next, it follows by applying Theorem 2 that F^n is a Riemannian space of constant curvature. Consequently Theorem 1 is proved.

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