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ON CERTAIN COMPLETE CONFORMALLY FLAT SPACES WITH CONSTANT SCALAR CURVATURE

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§ 0. Introduction.

Recently Yano, Houh and Chen ([5]) studied intrinsic problems of a conformally flat space in terms of sectional curvatures with respect to a unit vector field, in detail, they proved the following.

THEOREM 0. 1. *Let* (M, g) *be an n-dimensional Riemannian manifold with a unit vector field* u^h. *Then the necessary and sufficient conditions for* (M, g) *having the properties:*

(1) *The curvature operator* $K_{kji}{}^h v^k w^j$ *associated with two vectors* v^h *and* w^h *orthogonal to* u^h *annihilates* u^h;

(2) *Sectional curvature with respect to a section containing* u^h *is a constant*;

(3) *Sectional curvature with respect to a section orthogonal to* u^h *is a constant*;

are that the Riemann-Christoffel curvature tensor of (M, g) *has the form*

$K_{kji}{}^h=\lambda(\delta_k^h g_{ji}-\delta_j^h g_{ki})+\mu\{(\delta_k^h u_j-\delta_j^h u_k)u_i+(u_k g_{ji}-u_j g_{ki})u^h\}$

for some functions λ *and* μ, g_{ji} *being the Riemannian metric of* (M, g). *In this case,* (M, g) *is a conformally flat space for* $n>3$.

In the present paper, we investigate a conformally flat space with two mutually ortho gonal unit vector fields and with similar types of conditions of sectional curvatures as those stated in Theorem 0. 1. Our main result is appeared in Theorem 2. 4.

§ 1. Certain conformally flat spaces with two unit vector fields.

Let (M, g) be a Riemannian manifold with metric tensor g, whose components are g_{ji} with respect to local coordinate $\{\xi^h\}$, where, here and in the sequel the indicies $h, j, i, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. If we denote by $K_{kji}{}^h$ the Riemann-Christoffel curvature tensor of (M, g), then the Ricci tensor and the scalar curvature are given respectively by $K_{ji}=K_{tji}{}^t$, $K=g^{ji}K_{ji}$, where g^{ji} are contravariant components of g.

We define a tensor field L_{ji} of type (0, 2) by

$$L_{ji}=-\frac{1}{(n-2)}K_{ji}+\frac{K}{2(n-1)(n-2)}g_{ji}. \tag{1.1}$$

The Weyl conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h=K_{kji}{}^h+\delta_k^h L_{ji}-\delta_j^h L_{ki}+L_k{}^h g_{ji}-L_j{}^h g_{ki}, \tag{1.2}$$

where, $L_k{}^h=L_{kt}g^{th}$.

An n-dimensional Riemannian manifold (M, g) is conformally flat ([3], [4]) if and only if

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(1.3) $$C_{kji}{}^h=0 \quad \text{for } n>3,$$

(1.4) $$\nabla_k L_{ji}-\nabla_j L_{ki}=0 \qquad \text{for } n=3.$$

It is well known that (1.4) can be derived from (1.3) for $n>3$.

In this section we prove

PROPOSITION 1.1. *Let (M, g) be an n-dimensional conformally flat space $(n>3)$ with mutually orthogonal unit vector fields u^h and v^h satisfying the following two conditions:*

(I) The curvature operator $K_{kji}{}^h X^k Y^j$ associated with two vector fields X^h and Y^h orthogonal to u^h and v^h respectively annihilates u^h and v^h;

(1.5) $$K_{kji}{}^h X^k Y^j u^i=0, \qquad K_{kji}{}^h X^k Y^j v^i=0.$$

(II) Sectional curvature $K(\sigma)$ with respect to a section containing u^h orthogonal to v^h, and vice versa are same value, and $K(\sigma)$ is a constant. Then we have

$$L_{ji}=\alpha g_{ji}+\beta(u_j u_i+v_j v_i)$$

for some functions α and β.

Proof. We take n-2 linearly independent vectors $B_a{}^h$, $(a, b, c, \ =\{1, 2, 3, \ , n-2\})$, orthogonal to given unit vectors u^h and v^h and let $B^a{}_i, u_i, v_i$ be determined in such a way that $(B_a{}^h, u^h, v^h)^{-1}=(B^a{}_i, u_i, v_i)$. Then we have

(1.6) $$B_b{}^h B^b{}_i=\delta_i^h-u_i u^h-v_i v^h,$$

(1.7) $$B_c{}^i B^b{}_i=\delta_c^b, \qquad B_a{}^h u_h=0, \qquad B_b{}^h v_h=0.$$

The condition (I) is experssed as

(1.8) $$K_{kjih}B_c{}^k u^j v^i=0,$$

(1.9) $$K_{kjih}B_c{}^k v^j u^i=0,$$

(1.10) $$K_{kjih}B_c{}^k B_b{}^j u^i=0,$$

(1.11) $$K_{kjih}B_c{}^k B_b{}^j v^i=0.$$

Transvecting $B^c{}_m$ to (1.8) and using (1.6), we find

$$(\delta_m^k-u_m u^k-v_m v^k)K_{kjih}u^j v^i=0,$$

or, equivalently

(1.12) $$K_{mjih}u^j v^i=v_m K_{kjih}v^k u^j v^i,$$

from which, tranvecting with u^h,

(1.13) $$K_{mjih}u^j v^i u^h=-\mu v_m,$$

where $\mu=-K_{kjih}u^k v^j u^i v^h$.

Similarly, transvecting (1. 9) with $v^h B^c{}_m$, we have

$$K_{mjih} v^j u^i v^h = -\mu u_m. \tag{1.14}$$

Comparing with (1. 12) and (1. 14), we obtain

$$K_{mjih} u^j v^i = -\mu v_m u_h. \tag{1.15}$$

From the second part of condition (II), we have $K_{kjih} X^k u^j X^i u^h$=constant for any unit vector X^h orthogonal to u^h. This fact can be written as

$$K_{kjih} u^k u^h B_c{}^j B_b{}^i = K(\sigma) g_{cb}. \tag{1.16}$$

If we transvect $B^c{}_m B^b{}_l$ to (1. 16) and take account of (1. 6) and (1. 15), then the left hand side becomes

$$B^c{}_m B^b{}_l K_{kjih} u^k u^h B^j{}_c B^i{}_b = K_{kmlh} u^k u^h - \mu v_l v_m. \tag{1.17}$$

And consequently (1. 16) implies

$$K_{kjih} u^k u^h - \mu v_j v_i = K(\sigma)(g_{ji} - u_j u_i - v_j v_i) \tag{1.18}$$

because of (1. 6).

We can get from (1. 2) and (1. 3),

$$K_{kjih} = -L_{kh} g_{ji} + L_{ki} g_{jh} - L_{ji} g_{kh} + L_{jh} g_{ki}. \tag{1.19}$$

Transvecting (1. 19) with $u^j v^i$ and using (1. 15), we get

$$-\mu u_h v_k = -L(u, v) g_{kh} + (L_{kt} v^t) u_h + (L_{ht} u^t) v_k, \tag{1.20}$$

from which, transvecting with g^{kh},

$$L(u, v) = 0 \tag{1.21}$$

for $n>2$, where $L(u, v)$ is defined by $L_{ji} u^j v^i$. Thus (1. 20) implies that

$$L_{jt} u^t = L(u, u) u_j, \qquad L_{jt} v^t = L(v, v) v_j, \tag{1.22}$$

$$L(u, u) + L(v, v) = -\mu. \tag{1.23}$$

If we transvect $u^k u^h$ to (1. 19) and using (1. 22), then we obtain

$$K_{kjih} u^k u^h = -L_{ji} + L(u, u)(-g_{ji} + 2u_j u_i). \tag{1.24}$$

From (1. 18) and (1. 24), we may have

$$\begin{aligned} L_{ji} = &- \{K(\sigma) + L(u, u)\} g_{ji} \\ &+ \{K(\sigma) + 2L(u, u)\} u_j u_i + \{K(\sigma) - \mu\} v_j v_i. \end{aligned} \tag{1.25}$$

The first assumption of condition (II) means that

$$K_{kjih}B_c{}^jB_b{}^iu^ku^h=K_{kjih}B_c{}^jB_b{}^iv^kv^h,$$

from which, transvecting $B^c{}_mB^b{}_l$ and taking account of (1.17),

$$K_{kmlh}u^ku^h-\mu v_lv_m=B^c{}_mB^b{}_lK_{kjih}B_c{}^jB_b{}^iv^kv^h,$$

or. using (1.6) and (1.15),

$$K_{kmlh}u^k{}^h-\mu v_lv_m=K_{kmlh}v^kv^h-\mu u_mu_l.$$

Substituting (1.19) and (1.24) into the last equation and using (1.22), we have

$$-L_{ji}+L(u,u)(-g_{ji}+2u_ju_i)-\mu v_jv_i$$
$$=-L_{ji}+L(v,v)(-g_{ji}+2v_jv_i)-\mu u_ju_i,$$

from which, transvecting g^{ji} and using (1.23),

$$L(u,u)=L(v,v),\qquad 2L(u,u)=-\mu. \tag{1.26}$$

Thus (1.25) becomes

$$L_{ji}=\alpha g_{ji}+\beta(u_ju_i+v_jv_i). \tag{1.27}$$

where we have put

$$\alpha=\frac{1}{2}\mu-K(\sigma),\qquad \beta=K(\sigma)-\mu. \tag{1.28}$$

This completes the proof of the proposition.

§2. A theorem on a complete conformally flat space with constant scalar curvature.

In this section, we consider a complete conformally flat space with constant scalar curvature satisfying all assumptions as those stated in Proposition 1.1.

Substituting (1.27) into (1.19), we find

$$\begin{aligned}K_{kjih}=&2\alpha(g_{jh}g_{ki}-g_{kh}g_{ji})\\&+\beta\{g_{ki}(u_ju_h+v_jv_h)-g_{ji}(u_ku_h+v_kv_h)\\&+g_{jh}(u_ku_i+v_kv_i)-g_{kh}(u_ju_i+v_jv_i)\},\end{aligned} \tag{2.1}$$

from which transvecting with g^{kh},

$$K_{ji}=2\{(1-n)\alpha-\beta\}g_{ji}+(2-n)\beta(u_ju_i+v_jv_i), \tag{2.2}$$

and consequently

$$K=-2(n-1)(n\alpha+2\beta). \tag{2.3}$$

Transvecting (2. 2) with u^i and v^i respectively, we can easily see

$$K_{ji}u^t = \{2(1-n)\alpha - n\beta\} u_j, \qquad (2.4)$$
$$K_{ji}v^t = \{2(1-n)\alpha - n\beta\} v_j.$$

If we transvect (2. 2) with $K_k{}^i$ and using (2.2) and (2.4), then

$$K_{ji}K_k{}^t = \{4(1-n)\alpha - (n+2)\beta\} K_{jk} - 2\{2(n-1)\alpha + n\beta\}\{(n-1)\alpha+\beta\} g_{jk}, \qquad (2.5)$$

which implies that

$$K_{ji}K^{ji} = 2(n-1)(n\alpha+2\beta)\{4(n-1)\alpha+(n+2)\beta\} - 2n\{2(n-1)\alpha+n\beta\}\{(n-1)\alpha+\beta\} \qquad (2.6)$$

by virtue of (2. 3).
Transvecting (2.5) with K^{jk} and using (2.3) and (2. 6), we find

$$\begin{aligned} &K_{ji}K_k{}^tK^{jk} \\ &= \{4(1-n)\alpha - (n+2)\beta\} [2(n-1)(n\alpha+2\beta) \\ &\times \{4(n-1)\alpha + (n+2)\beta\} - 2n\{2(n-1)\alpha+n\beta\} \times \{(n-1)\alpha+\beta\}] \\ &+ 4(n-1)(n\alpha+2\beta)\{(n-1)\alpha+\beta\}\{2(n-1)\alpha+n\beta\}. \end{aligned} \qquad (2.7)$$

We see from (2. 5) that the eigenvalue λ of $K_j{}^h$ satisfies the quadratic equation

$$\lambda^2 + \{4(n-1)\alpha + (n+2)\beta\}\lambda + 2\{2(n-1)\alpha+n\beta\}\{(n-1)\alpha+\beta\} = 0,$$

that is,

$$\lambda_1 = -2(n-1)\alpha - n\beta, \qquad \lambda_2 = -2(n-1)\alpha - 2\beta. \qquad (2.8)$$

LEMMA 2.1. *In a conformally flat space of dimension n with constant scalar curvature K, we have* (*cf. See* [6]).

$$\begin{aligned} &\frac{1}{2}\Delta(K_{ji}K^{ji}) = \\ &\frac{n}{n-2}K_t{}^sK_s{}^rK_r{}^t - \frac{2n-1}{(n-1)(n-2)}K\, K_{ji}K^{ji} \\ &+ \frac{1}{(n-1)(n-2)}K^3 - (\nabla_jK_{ih})(\nabla^jK^{ih}). \end{aligned} \qquad (2.9)$$

Substituting (2. 3), (2. 6) and (2. 7) into (2. 9), we find the relationship

$$\frac{1}{2}\Delta(K_{ji}K^{ji})=\frac{n}{n-2}[\{4(1-n)\alpha-(n+2)\beta\}(2(n-1)(n\alpha+2\beta)$$
$$\times\{4(n-1)\alpha+(n+2)\beta\}-2n\{2(n-1)\alpha+n\beta\}\{(n-1)\alpha+\beta\})$$
$$+4(n-1)(n\alpha+2\beta)\{(n-1)\alpha+\beta\}\{2(n-1)\alpha+n\beta\}]$$
$$-\frac{2n-1}{(n-1)(n-2)}\{-2(n-1)(n\alpha+2\beta)\}[2(n-1)(n\alpha+2\beta)$$
$$\times\{4(n-1)\alpha+(n+2)\beta\}-2n\{2(n-1)\alpha+n\beta\}\{(n-1)\alpha+\beta\}]$$
$$+\frac{1}{(n-1)(n-2)}\{-2(n-1)(n\alpha+2\beta)\}^3+(\nabla_jK_{ih})(\nabla^jK^{ih})$$

or, equivalently

$$\frac{1}{2}\Delta(K_{ji}K^{ji})=-2(n-2)^3(2\alpha+\beta)\beta^2-(\nabla_kK_{ji})(\nabla^kK^{ji}). \tag{2.10}$$

LEMMA 2.2. *Let (M,g) be a conformally flat space of dimension $n>3$ such that conditions (I) and (II) of Proposition 1.1 are satisfied, and the scalar curvatnre K is constant, then α and β are both constants on (M,g).*

Proof. Differentiating (1.27) covariantly, we have

$$\nabla_kL_{ji}=(-\frac{2}{n}g_{ji}+u_ju_i+v_jv_i)\beta_k$$
$$+\beta\{(\nabla_ku_j)u_i+(\nabla_kv_j)v_i+u_j(\nabla_ku_i)+v_j(\nabla_kv_i)\}, \tag{2.11}$$

where $\beta_k=\nabla_k\beta$.

On the other hand, we have from (1.1)

$$\nabla_tL_k^{\ t}=0 \tag{2.12}$$

because of K=constant.

Transvecting (2.11) with g^{jk} and using (2.12), we have

$$\{-\frac{2}{n}\beta_i+(u^t\beta_t)u_i+(v^t\beta_t)v_i\}$$
$$+\beta\{u_i(\nabla_tu^t)+v_i(\nabla_tv^t)+u^t(\nabla_tu_i)+v^t(\nabla_tv_i)\}=0. \tag{2.13}$$

Taking skew-symmetric parts of (2.11) with respect to k and j and making use of (1.4), we obtain

$$(-\frac{2}{n}g_{ji}+u_ju_i+v_jv_i)\beta_k-(-\frac{2}{n}g_{ki}+u_ku_i+v_kv_i)\beta_j$$
$$+\beta\{(\nabla_ku_j-\nabla_ju_k)u_i+(\nabla_kv_j-\nabla_jv_k)v_i$$
$$+u_j\nabla_ku_i-u_k\nabla_ju_i+v_j\nabla_kv_i-v_k\nabla_jv_i\}=0 \tag{2.14}$$

from which, transvecting with $u^j u^i$ and $v^j v^i$ respectively,

(2.15) $$\frac{n-2}{n}\{\beta_k-(u^t\beta_t)u_k\}+\beta\{-u^t\nabla_t u_k-u^t u^s(\nabla_t v_s)v_k\}=0,$$

(2.16) $$\frac{n-2}{n}\{\beta_k-(v^t\beta_t)v_k\}+\beta\{-v^t\nabla_t v_k-v^t v^s(\nabla_t u_s)u_k\}=0.$$

Transvecting (2.15) with v^k and (2.16) with u^k, we have respectively

(2.17) $$v^t\beta_t=0, \qquad u^t\beta_t=0.$$

Substituting (2.15) and (2.16) into (2.13) and taking account of (2.17), we find

(2.18) $$\frac{2(n-3)}{n}\beta_k=Au_k+Bv_k$$

for suitable functions A and B, and consequently β is constant by virtue of (2.17).

Since K and β are both constants, α is also because of (2.3). Thus Lemma 2.2 is proved.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma 2.2, we have* $\beta=0$ *or* $2\alpha+\beta=0$ *on* (M,g).

Proof. We have from (2.5) and Lemma 2.2

(2.19) $$K_{jt}K_i^{\ t}=aK_{ji}+bg_{ji}$$

for some constants a and b, where

(2.20) $$a=4(1-n)\alpha+(n+2)\beta,$$

$$b=-2\{2(n-1)\alpha+n\beta\}\{(n-1)\alpha+\beta\}.$$

On the other hand, we see that

(2.21) $$\nabla_k K_{ji}-\nabla_j K_{ki}=0$$

because of (1.1), (1.4) and K=constant.

Differentiating (2.19) covariantly, we get

(2.22) $$(\nabla_k K_{jt})K_i^{\ t}+K_{jt}(\nabla_k K_i^{\ t})=a\nabla_k K_{ji},$$

from which, taking skew-symmetric parts with respect to k and j and using (2.21),

$$K_{jt}(\nabla_k K_i^{\ t})-K_{kt}(\nabla_j K_i^{\ t})=0,$$

or, changing k with i,

(2.23) $$K_{jt}(\nabla_i K_k^{\ t})-K_{it}\nabla_j K_k^{\ t}=0.$$

Adding (2.22) to (2.23), we find

$$2K_{jt}(\nabla_i K_k^{\ t})=a\nabla_k K_{ji},$$

from which, transvecting $K_l{}^i$ and using (2.19),

$$aK_{lt}\nabla_i K_k{}^t + 2b\nabla_i K_{lk} = 0.$$

Thus the last two equations mean that

$$(a^2-4b)\nabla_k K_{ji} = 0.$$

Since a and b are both constants, we have $a^2-4b=0$ or $\nabla_k K_{ji}=0$ on (M, g). If $a^2-4b=0$, then we easily verify that $\beta=0$ because of (2.8) and (2.20).

If $\nabla_k K_{ji}=0$, then $\Delta(K_{ji}K^{ji})=0$. In this case, we see $\beta=0$ or $2\alpha+\beta=0$ by virtue of (2.10). This complete the proof.

Summing up the arguments developed above, if $\beta=0$, (M, g) is a real space form because of (2.1). If $2\alpha+\beta=0$, then the eigenvalues of the Ricci tensor $K_j{}^h$ are 2α or $-2\alpha(n-3)$ by virtue of (2.8). In usual way (cf. See Ryan [1], Sekigawa and Tagagi [2]), owing to completeness, (M^n, g) is one of E^n or $E^2\times S^{n-2}(c)$.

We are now in a position to have a theorem.

THEOREM 2.4. *Let* (M^n, g) $(n>3)$ *be a complete conformally flat space with mutually orthogonal unit vector fields* u^h *and* v^h *and with constant scalar curvature satisfying the two conditions*:

(*I*) *The curvature operator* $K_{kji}{}^h X^k Y^j$ *associated with two vector fields* X^h *and* Y^h *orthogonal to* u^h *and* v^h *respectively annihilates* u^h *and* v^h.

(*II*) *Sectional curvature* $K(\sigma)$ *with respect to a section containing* u^h *orthogonal to* v^h, *and vice versa are same value, and* $K(\sigma)$ *is a constant. Then* (M^n, g) *is one of* $S^n(c)$, E^n *or* $E^2\times S^{n-2}(c)$, *the real space forms of curvature* c *being denoted by* $S^n(c)$ *or* E^n *depending on whether* c *is positive or zero.*

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