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RESTRICTED DOUBLE AUTOMORPHISMS OF THE SPACE OF ANALYTIC DIRICHLET FUNCTIONS

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1. (**a**) **Introduction**. Let $\boldsymbol{C}$ be the complex plane equipped with its usual topology. Let X be the family of all Dirichlet functions with abscissa of convergence and absolute convergence greater than or equal to $A>0$ (see [4], page 33). For each $f\in X$, define

$$p(\sigma,f)=\sum_{n=1}^{\infty}|a_n|e^{\sigma\lambda_n},\quad \text{where } f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n},\quad s=\sigma+it\in \boldsymbol{C},$$

where σ is arbitrary and $<A$. Clearly, this defines a semi-norm on X. Denote by $(X, \mathcal{T})$, the space X equipped with the locally convex topology $\mathcal{T}$ generated by the family $\{p(\sigma,\cdots):\sigma<A\}$ of semi-norms. We consider another space $(Y, \mathcal{Q})$, $Y\subset X$, equipped with a certain Fréchet topology $\mathcal{Q}$ which is stronger than the topology induced on Y by $\mathcal{T}$. The main aim of this paper is to construct restricted double automorphisms (see definition below) on X and Y. Throughout we assume that X and Y stand for locally convex spaces mentioned just now unless something else is stated regarding them.

If X is a topological vector space and Y is its subspace equipped with a topology stronger than the induced topology on Y from X, then an *automorphism* T on X and Y means a linear homeomorphic mapping of X onto itself *while a restricted double automorphism* on X and Y is a mapping T such that T is an automorphism on X and $T|Y$ (restriction of T on Y) is an automorphism on Y.

A base in the space X is a sequence α_n in X such that every element f in X is uniquely represented as follows:

$$f=\sum_{n=1}^{\infty}a_n\alpha_n, \tag{1.1}$$

where $\{a_n\}\subset \boldsymbol{C}$. A basis $\{\alpha_n\}$ in X is said to be *proper base* if for all sequences $\{a_n\}$ of complex numbers

$$\sum_{n=1}^{\infty}a_n\alpha_n \text{ converges in } X \Longleftrightarrow \sum_{n=1}^{\infty}a_n\delta_n \text{ converges in } X,$$

where $\delta_n(s)=e^{s\lambda_n}$. A characterization of a proper base has already been established by us in our paper [3] in terms of the following conditions:

$$(\alpha)\qquad \limsup_{n\to\infty}\frac{\log p(\sigma,f)}{\lambda_n}<A, \text{ for all } \sigma<A;$$

and

$$(\beta)\qquad \lim_{\sigma\to A}\{\liminf_{n\to\infty}\frac{\log p(\sigma,f)}{\lambda_n}\}\geq A.$$

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In the construction of a continuous linear map on X that we have in our mind, we shall need the following result which we have proved elsewhere in [2]:

LEMMA 1. *A necessary and sufficient condition that there exists a continuous linear transformation* $T: X \to X$ *with* $T\delta_n = \alpha_n$, $n=1, 2, \cdots$ *is that the condition* (α) *holds.*

1. (b) Construction of continuous linear map on X.

We now proceed to construct a continuous linear map on X. Let $\{\alpha_n\}$ be a proper base for X and $\{\phi_n\} \subset X$ be any sequence satisfying

$$(\alpha') \qquad \limsup_{n\to\infty} \frac{\log p(\sigma, \phi_n)}{\lambda_n} < A, \text{ for all } \sigma < A.$$

Then for each $f \in X$, there exists a unique sequence $\{a_n\}$ of complex numbers such that

$$\sum_{n=1}^{N} a_n \alpha_n \to f \text{ in } X, \text{ as } N \to \infty,$$

and

$$(1.2) \qquad \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \le -A.$$

Now by (α'), given $\sigma < A$, there exists $\varepsilon > 0$ and an integer $N_1 = N_1(\sigma, \varepsilon)$ such that

$$p(\sigma, \phi_n) \le e^{(A-\varepsilon)\lambda_n}, \text{ for all } n \ge N_1$$

$$\Longrightarrow \sum_{n=1}^{\infty} |a_n| p(\sigma, \phi_n) \text{ converges in } (X, \mathfrak{T}) \text{ for each } \sigma < A.$$

Hence $\sum_{n=1}^{\infty} a_n \phi_n$ is absolutely convergent and so convergent in $(X, \mathfrak{T})$. Thus with the above conditions imposed on $\{\alpha_n\}$ and $\{\phi_n\}$, we can define a mapping $P: X \to X$, as follows.

$$(1.3) \qquad P(f) = \sum_{n=1}^{\infty} a_n \phi_n, \quad f = \sum_{n=1}^{\infty} a_n \alpha_n.$$

It is seen that P is continuous, for taking into account (α'), there exists a $\sigma_1 < A$ such that

$$p(\sigma, \phi_n) < e^{\sigma_1 \lambda_n}, \text{ for all } n \ge N,$$

and hence there exists a constant $k > 0$, such that

$$p(\sigma, \phi_n) \le k e^{\sigma_1 \lambda_n}, \text{ for all } n \ge 1$$

$$= k p(\sigma_1, \delta_n), \text{ for all } n \ge 1.$$

Now, since $\{\alpha_n\}$ and $\{\delta_n\}$ are proper bases for X,

$$\sum_{n=1}^{\infty} a_n \delta_n \longleftrightarrow \sum_{n=1}^{\infty} a_n \alpha_n$$

is a topological isomorphism (see lemma 1 and Theorem 2. 2, [2]). Hence there exist constants k_2, M and numbers σ_2 and σ_3, such that

$$p(\sigma_1, \delta_2) \leq k_2 p(\sigma_2, \alpha_n); \text{ and } p(\sigma_2, \alpha_n) \leq M p(\sigma_3, \delta_n).$$

Hence

$$p(\sigma, \phi_n) \leq k p(\sigma_2, \alpha_n), k = k_1 k_2$$

$$\Longrightarrow \|pf\| \leq k \sum_{n=1}^{\infty} |a_n| p(\sigma_2, \alpha_n)$$

$$= k \sum_{n=1}^{\infty} p(\sigma_2, a_n \alpha_n) = kM \sum_{n=1}^{\infty} p(\sigma_3, a_n \delta_n)$$

$$= kMp(\sigma_3, f)$$

$\Longrightarrow$ p is continuous.

2. Continuity condition for mapping of X into Y.

In this section our concern lies in determining when the map P becomes a continuous linear map from $(X, \mathfrak{T})$ into $(Y, \mathcal{Q})$. Let us assume that $\{\|\cdot\|_\nu, \nu = 1, 2, \cdots\}$ stands for the family of semi-norms which generate the topology.

THEOREM 1. *A necessary and sufficient condition for P to be a continuous linear map from $(X, \mathfrak{T})$ into $(Y, \mathcal{Q})$ is that all ϕ_n belongs to Y and*

$$\limsup_{n\to\infty} \frac{\log \|\phi_n\|_\nu}{\lambda_n} < A \quad (\nu = 1, 2, \cdots). \tag{2. 1}$$

The expansion in (1. 3) *then converges in Y for all $f \in X$.*

Proof. Let $\phi_n \in Y$ and (2. 1) hold good. Then there exists $\varepsilon > 0$, such that

$$\|\phi_n\|_\nu < e^{(A-\varepsilon)\lambda_n},$$

for all $n \geq N$. We can find a constant k such that

$$\|\phi\|_\nu \leq k e^{((A-\varepsilon)\lambda_n}, \text{ for all } n \geq 1. \tag{2. 2}$$

Now any $f \in X$ can be represented as

$$f = \sum_{n=1}^{\infty} a_n \alpha_n,$$

where $\{\alpha_n\}$ is a proper base for X. Therefore by (1. 2), choosing $\delta < \varepsilon$, it follows that

$$|a_n| \leq e^{(-A+\delta)\lambda_n}, \text{ for all } n \geq N_1.$$

Hence the series $\sum_{n=1}^{\infty} a_n \phi_n$ converges (indeed absolutely) in $(Y, \mathcal{Q})$. Hence the map P : $(X, \mathfrak{T}) \to (Y, \mathcal{Q})$ is well defined. Let $\|\cdot\|_\nu$ be an arbitrary given but fixed semi-norm on Y. Then for this ν, we have by (2. 2), for $f \in X$

$$\|Pf\|_\nu \le k \sum_{n=1}^{\infty} |a_n| e^{(A-\varepsilon)\lambda_n}$$

$$= kp(A-\varepsilon, f).$$

Hence P is a continuous linear operator from $(X. \mathcal{T})$ into $(Y, \mathcal{Q})$.

Conversely, assume that P is a continuous linear operator from $(X, \mathcal{T})$ into $(Y, \mathcal{Q})$. Clearly then $\{\phi_n\} \subset Y$. Consider any sequence $\{a_n\} \subset \boldsymbol{C}$, such that, $\sum_{n=1}^{\infty} a_n\alpha_n$ converges in $(X, \mathcal{T})$.

$$\Longrightarrow a_n\alpha_n \to 0 \text{ in } (X, \mathcal{T})$$

$$\Longrightarrow P(a_n\alpha_n) \to 0 \text{ in } (Y, \mathcal{Q}).$$

Hence

(2. 3) $$|a_n| \|\phi_n\|_\nu \to 0 \text{ as } n \to \infty, \text{ for each } \nu = 1, 2, \cdots.$$

Suppose (2. 1) is not true. Then for some semi-norm $\|\cdot\|_\nu$, there exists a sequence $\{r_l\}$, with $r_1 < r_2 < \cdots < r_l \to A$ as $l \to \infty$ such that

(2. 4) $$\frac{\log \|\phi_{n_l}\|_\nu}{\lambda_{n_l}} > r_l.$$

Define a sequence $\{a_n\}$ of complex numbers as follows:

$$a_n = \begin{cases} \dfrac{1}{\|\phi\|_\nu}; & n = n_l, \quad l = 1, 2, \cdots \\ 0 \;; & n \ne n_l, \quad l = 1, 2, \cdots. \end{cases}$$

Then from (2. 4) and this choice of a_n, it follows that

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \le -A$$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n\alpha_n \text{ converges in } (X, \mathcal{T})$$

$$\Longrightarrow |a_n| \|\phi\|_\nu \to 0 \text{ as } n \to \infty \text{ in } (Y, \mathcal{Q}), \text{ by (2. 3)}.$$

But this is contradicted by the fact that $|a_n| \|\phi_n\|_\nu = 1$, for $n = n_l$. This completes the proof.

If we restrict the class Y, then a simpler condition for P to be continuous linear map can be established as stated in the following theorem.

THEOREM 2. *Suppose Y consists for all functions $g(\in X)$ of the form $g = \sum_{n=1}^{\infty} a_n\phi_n$ for which $\sup_{\sigma<A} [p(\sigma, \phi_n)] < \infty$, and the topology on Y is weaker than that determined by the sup norm. If the functions $\phi_n (n=1, 2, \cdots)$ belong to Y and are uniformly continuous in the* *half-plane $\sigma<A$, then P is a continuous linear mapping from X into Y.*

Proof. Let P_μ $(-\infty<\mu<0)$ be a mapping from X into Y defined by

$$(P_\mu f)(\sigma+it)=Pf(\mu+\sigma+it),\quad \sigma<A.$$

Thus

$$(P_\mu f)(\sigma+it)=\sum_{n=1}^{\infty} a_n\phi_n^\mu(\sigma+it),$$

where

$$\phi_n^\mu(\sigma+it)=\phi_n(\sigma+\mu+it),\quad \sigma<A,\quad n=1,2,\cdots.$$

From the condition (α') we get,

$$\limsup_{n\to\infty}\frac{\log\|\phi_n^\mu\|_\mu}{\lambda_n}<A.$$

In fact, since the topology on Y is weaker than that induced by the sup norm, therefore given any ν, there exists a constant k, such that

$$\|\phi_n^\mu\|_\nu\leq k\|\phi_n^\mu\|,$$

where

$$\|\phi_n^\mu\|=\sup_{\sigma<A}\{\sup_{-\infty<t<\infty}|\phi^\mu(\sigma+it)|\}.$$

Hence given $\varepsilon<0$, there exists a $\sigma<A$, such that

$$\begin{aligned}\|\phi_n^\mu\| &< \sup_{-\infty<t<\infty}|\phi_n^\mu(\sigma+it)|+\varepsilon\\ &\leq p(\sigma+\mu,\phi_n)+\varepsilon\\ &\leq e^{(A+\mu)\lambda_n}+\varepsilon\\ &=e^{(A+\mu)\lambda_n}\{1+0(1)\}\\ \Longrightarrow\ &\limsup_{n\to\infty}\frac{\log\|\phi_n\|_\nu}{\lambda_n}<A,\end{aligned}$$

for $\nu\geq1$, and for each μ, $-\infty<\mu<0$. This, by Theorem 1, implies that P_μ maps X continuously into Y. Clearly the family P_μ $(-\infty<\mu<0)$ is pointwise bounded, since $\|P_\mu f\|\leq\|Pf\|$, for all μ and each $f\in X$. Hence by Banach-Steinhaus Theorem ([1], p. 55, Theorem 18), this family is uniformly bounded. Moreover the uniform continuity of ϕ_n implies that

$$\lim_{\mu\to0}|\phi_n(\mu+z)-\phi_n(z)|=0,\quad z=\sigma+it,\quad \sigma<A.$$

$$\Longrightarrow\quad \lim_{\mu\to0}|P_\mu\delta_n(\sigma+it)-P\delta_n(\sigma+it)|=0$$

Hence

$$\lim_{\mu\to 0} \|P_\mu \delta_n - P\delta_n\| = \lim_{\mu\to 0\delta} \{\sup_{\sigma<A} \sup_{-\infty<t<\infty} |P_\mu\delta_n(\sigma+it) - P\delta_n(\sigma+it)|\} = 0, \quad n=1, 2, \cdots,$$

i. e., $\{P_\mu\}$ converges to P on a total subset of X. Hence P is a continuous linear mapping of X into Y.

3. Construction of restricted double automorphisms.

In this section we confine our attention to the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in X for which the function

$$\phi_n = \beta_n - \alpha_n, \tag{3.1}$$

belongs to Y and satisfies (2. 1). Then corresponding to any prescribed seminorm $\|\cdot\|_\nu$ on Y, there exists a number ρ, such that

$$\limsup_{n\to\infty} \frac{\log\|\phi_n\|_\nu}{\lambda_n} \leq \rho < A, \tag{3.2}$$

holds. Since the topology on Y is stronger than that induced by X on Y, to each given $\sigma<A$, there corresponds a constant K and a positive integer ν, such that

$$p(\sigma, f) \leq k\|f\|_\nu, \text{ for all } f\in Y. \tag{3.3}$$

From (3. 2) and (3. 3), it follows that for any $\sigma<A$, there are positive constants M and $\rho<A$, such that

$$p(\sigma, \phi_n) \leq Me^{\rho\lambda_n}, \quad n=1, 2, \cdots. \tag{3.4}$$

In view of these observations, we prove that following result.

LEMMA 2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in X for which the function ϕ_n of* (3. 1) *belongs to Y $(n=1, 2, \cdots)$ and satisfies* (2. 1). *Then the sequence $\{\beta_n\}$ satisfies the condition (α) if and only if $\{\alpha_n\}$ does.*

Proof. Let us assume that $\{\alpha_n\}$ satisfies the condition (α). Then there exists a constant $\rho_1<A$, such that

$$\limsup_{n\to\infty} \frac{\log p(\sigma, \alpha_n)}{\lambda_n} < \rho_1,$$

we also have

$$p(\sigma, \beta_n) \leq p(\sigma, \alpha_n) + p(\sigma, \phi_n), \text{ for all } \sigma<A.$$

By (3. 3) given σ, there exists a ν and a constant k such that

$$p(\sigma, \phi_n) \leq k\|\phi_n\|_\nu.$$

By (2. 1), given ν, there exists a constant $\rho_2<A$, such that

$$\limsup_{n\to\infty} \frac{\log\|\phi_n\|_\nu}{\lambda_n} < \rho_2 < A.$$

$$\Longrightarrow \quad \|\phi_n\|_\nu \le e^{\rho_2\lambda_n}, \text{ for all } n \ge N$$

$$\Longrightarrow \quad p(\sigma, \phi_n) \le k e^{\rho_2\lambda_n}, \text{ for all } n \ge N.$$

Choose $\rho = \max(\rho_1, \rho_2)$, then

$$p(\sigma, \beta_n) \le e^{\rho\lambda_n} + k e^{\rho\lambda_n} = (k+1) e^{\rho\lambda_n}$$

$$\Longrightarrow \quad \limsup_{n\to\infty} \frac{\log p(\sigma, \beta_n)}{\lambda_n} \le \rho < A,$$

and so $\{\beta_n\}$ satisfies (α). Hence the result follows by the symmetry of the given condition.

In the statement of the above lemma, if we replace the condition (α) by (β), then the result is not necessarily true. For example, when $Y=X$ then $\mathscr{C}=\mathscr{Q}$. Consider then

$$-\alpha_n(s) = \phi_n(s) = e^{s\lambda_n}, \quad n=1, 2, \cdots.$$

If Y is taken to be a Banach space, then the above assertion is valid. In this connection, we prove the following lemma.

LEMMA 3. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* X *for which the function* ϕ_n *of* (3. 1) *belongs to* $Y(n=1, 2,)$ *and satisfies*

$$\sup_{\nu\ge1} \{\limsup_{n\to\infty} \frac{\log \|\phi_n\|_\nu}{\lambda_n}\} < A. \tag{3.5}$$

Then the sequence $\{\beta_n\}$ *satisfies condition* (β) *if and only if* $\{\alpha_n\}$ *does.*

Proof. From hypothesis (3. 5), it follows that we can find a number $\rho < A$ such that

$$\limsup_{n\to\infty} \frac{\log \|\phi_n\|_\nu}{\lambda_n} < \rho,$$

for all $\nu \ge 1$, In view of (3. 3), this in turn implies that for each $\sigma < A$,

$$\limsup_{n\to\infty} \frac{\log p(\sigma, \phi_n)}{\lambda_n} < \rho.$$

Let us now assume that $\{\alpha_n\}$ satisfies condition (β) and λ be any number such that $\rho < \lambda < A$. For σ sufficiently near to A, we have then

$$\liminf_{n\to\infty} \frac{\log p(\sigma, \alpha_n)}{\lambda_n} > \lambda.$$

Also relation (3. 1) implies

$$p(\sigma, \beta_n) \ge p(\sigma, \alpha_n) - p(\sigma, \phi_n)$$

$$>e^{\lambda\lambda_n}-e^{\rho\lambda_n}, \text{ for all } n\geq \text{Max } (N_1, N_2)$$

$$=e^{\lambda\lambda_n}\ (1-e^{(\rho-\lambda)\lambda_n})$$

$$\Longrightarrow \liminf_{n\to\infty} \frac{\log p(\sigma, \beta_n)}{\lambda_n}\geq A$$

$$\Longrightarrow \lim_{\sigma\to A}\left\{\liminf_{n\to\infty}\frac{\log p\ (\sigma, \beta_n)}{\lambda_n}\right\}\geq A$$

Hence, $\{\beta_n\}$ satisfies condition (β) and the other part of the lemma follows by symmetry.

Since condition (α) and (β) are necessary and sufficient for a basis in X to be proper, lemma 2 and lemma 3 gives rise to the following theorem.

THEOREM 3. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be bases in* X *for which the function* $\{\phi_n\}$ *of* (3.1) *belongs to* Y $(n=1, 2, \cdots)$ *and satisfies* (3.5). *Then for* $\{\beta_n\}$ *to be proper, it is necessary and sufficient that* $\{\alpha_n\}$ *be proper.*

Now our aim is to define restricted double automorphisms on X and Y. For this, we first state the following simple result, whose proof follows from the open mapping theorem ([1], p. 57).

LEMMA 4. *Let* $T=S+P$, *where* S *is a restricted double automorphism on* X *and* Y *and* P *is a continuous linear mapping of* X *into* Y. *If* T *is an automorphism on* X, *then* T *is, in fact, a restricted double automorphism on* X *and* Y.

THEOREM 4. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be proper bases in* X *and let* T *be the endomorphism mapping* $\{\alpha_n\}$ *on to* $\{\beta_n\}$. *If the function* $\phi_n=\beta_n-\alpha_n$ *belongs to* Y $(n=1, 2, \cdots)$ *and satisfies the condition*

$$\limsup_{n\to\infty} \frac{\log \|\phi_n\|_\nu}{\lambda_n}<A, \quad (\nu=1, 2, \cdots),$$

then T *is a restricted double automorphism on* X *and* Y.

Proof: Let for any function $f\in X$, its expansion in the basis $\{\alpha_n\}$ be given by

$$f=\sum_{n=1}^{\infty}a_n\alpha_n$$

Then Tf is given by

$$Tf=\sum_{n=1}^{\infty}a_n\beta_n$$

$$=\sum_{n=1}^{\infty}a_n\alpha_n+\sum_{n=1}^{\infty}a_n\phi_n.$$

If we denote the identity map by I, then $T=I+P$, where P is defined as in (1.4). But by theorem 1, P maps X continuously into Y and I is obviously a restricted double automorphism on X and Y. Using lemma 4, T becomes a restricted double automorphism on X and Y. This completes the proof.

The following result immediately follows from Theorem 3.

COROLLARY 4.1. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be bases in* X *for which the function*

$$\phi_n = \beta_n - \alpha_n$$

belongs to Y $(n=1, 2, \cdots)$ *and satisfies the condition*

$$\sup_{\nu \geq 1} \{\lim_{n\to\infty} \sup \frac{\log\|\phi_n\|_\nu}{\lambda_n}\} < A.$$

If one of the given bases is proper, then both are proper, and the endomorphism T *mapping* $\{\alpha_n\}$ *onto* $\{\beta_n\}$ *is a restricted double automorphism on* X *and* Y.

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