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ON QUASI-SEMIDEVELOPABLE SPACES

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1. Introduction.

In recent years there have been several studies concerning the generalization of developable spaces. H. R. Bennett [3] defines a quasi-developable space which is useful to obtaining metrization theorems for M_i-spaces ($i=1,2,3$). C. C. Alexander [1] introduced semi-developable spaces and proves that a space is semi-metrizable if and only if it is semi-developable T_0-space. He also introduces cushioned pair semidevelopable spaces [2] to obtain a generalization of Morita's metrization theorem.

In the paper [8] we have generalized developable spaces further and introduced a quasi-semidevelopable space which includes both classes of semi- and quasi-developable spaces. We denote a quasi-semidevelopable space by a qs-developable space as in [8]. Thus we have shown that

(1) A space is semi-developable if and only if it is qs-developable and perfect.

(2) In a qs-developable space hereditary χ_1-compactness, hereditary Lindelöf property and hereditary separability are equivalent,

(3) A separable regular T_0-space with a point-finite qs-development is metrizable.

In the present paper we extend some of the results appeared in [8]. A qs-stratifiable space is defined and show that it is semi-stratifiable if it is perfect. The closure preserving property for the qs-developable space is investigated to find relations between qs-developable spaces and M_i-spaces. By doing so, we get an example which shows that a regular closure preserving semi-developable space is not always metrizable. We give more general definition of cushioned pair qs-development than the one appeared in [8]. Thus we obtain a generalization of theorem 2.3 of [8].

By a space we will mean a topological space in this paper. We assume every topological space is T_1 unless otherwise mentioned. We adopt the convention if G is a subset of a topological space X, then $\mathrm{Int}(G)$ denotes the interior of G in X and $\mathrm{cl}(G)$ denotes the closure of G in X. If G is a collection of sets, then $G^*=\bigcup\{g\mid g\in G\}$. Finally N denotes the set of all positive integers. All undefined terms are as in [7].

2. Quasi-semidevelopable spaces.

Let $\gamma=(\gamma_1,\gamma_2,\cdots)$ be a sequence of collections of subsets of a topolgical space (X,τ). Consider following three conditions of the sequence γ:

(1) For each $x\in X$ $\{St(x,\gamma_n)\mid n\in N,\ x\in\gamma_n{}^*\}$ is a local basis at x.

(2) Each γ_n is a covering of X.

(3) Each γ_n is a subclass of τ.

The condition (1) is equivalent to the following:

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(a) For each $x\in X$ and for each positive integer n $St(x,\gamma_n)$ is a neighborhood of x provided $St(x,\gamma_n)\neq\phi$, and

(b) For each $x\in X$ and for each open set U containing x there exists a positive integer n such that $x\in St(x,\gamma_n)\subset U$.

If γ satisfies all the above three conditions (1), (2) and (3), then γ is called a *development* for the space X. And γ is a *semi-development* for X if it satisfies only the two conditions (1) and (2) [1]. On the other hand if γ satisfies the conditions (1) and (3), then it is called a *quasi-development* for X [3].

We generalize these spaces and define a new class of spaces.

DEFINITION 2.1. A sequence $\gamma=(\gamma_1,\gamma_2,\cdots)$ of collections of subsets of a space X is a *quasi-semidevelopment* for X if γ satisfies the condition (1).

A space is said to be quasi-semidevelopable if it has a quasi-semidevelopment.

From the definition it is clear that semi-developable spaces and quasi-developable spaces are qs-developable.

G.D. Creede [6] introduced a class of semi-stratifiable spaces as a generalization of semi-metric spaces. He shows that a T_1-space is semi-metric if and only if it is first countable and semi-stratifiable. The class of semi-stratifiable spaces contains M_3-spaces [4]. (For the definition of M_3-spaces see section 3.) Now we introduce a class of qs-stratifiable spaces as a generalization of qs-developable spaces and show that a qs-stratifiable space is semi-stratifiable if it is perfect.

LEMMA 2.2. *A space X is T_1 and qs-developable if and only if there is a mapping g: $N\times X\to\mathcal{P}(X)$ such that*

(1) *the set $z_x=\{n\in N|g(n,x)\neq\phi\}$ is infinite,*

(2) *the collection $\{g(n,x)|n\in Z_x\}$ is a local basis at the point x,*

(3) *for each $i,k\in Z_x$ $g(i,x)\subset g(k,x)$ if $i<k$,*

(4) *if $x\in g(n,x_n)$ for every $n\in Z_x$, then x is a cluster point of the sequence $\langle x_n\rangle$, and*

(5) *if $x\in g(n,y)$, then $x\in g(n,x)$.*

Proof. Let $\gamma=(\gamma_1,\gamma_2\cdots\cdots)$ be a qs-development for the space X. We may assume that the set $\{n|x\in\gamma_n{}^*\}$ is infinite for each point of X. Let $f: N\times X\to P(X)$ be the mapping such that $f(n,x)=St(x,\gamma_n)$ and $z'_x=\{n\in N|f(n,x)\neq\phi\}$. Then clearly the set Z'_x is infinite. In order to get a mapping which satisfies the above conditions we define a mapping g as follows:

$$g(n,x)=\left|\begin{array}{ll}\bigcap\limits_{\substack{i\leq n\\ i\in Z'_x}}f(i,x) & \text{if } n\in Z'_x\\ \phi & \text{if } n\overline{\in} Z'_x.\end{array}\right.$$

The set $Z_x=\{n\in N|g(n,x)\neq\phi\}$ is infinite since $Z_x=Z'_x$. Clearly $\{f(n,x)|n\in N$ and $x\in\gamma_n{}^*\}$ is a local basis at x for each $x\in X$, and so is $\{g(n,x)|n\in Z_x\}$. The third property of the lemma is clear.

To show (4), let $x\in g(n,x_n)$ for every $n\in Z_x$. Then it is easily seen that x_n is in every $g(n,x)$ for $n\in Z_x$. Since the sequence of sets $\{g(n,x)\}$ is decreasing, x is a cluster point of the sequence $<x_n>$.

To show the last property, let $x \in g(n, y)$. Then $x \in \bigcap_{\substack{i \leq n \\ i \in Z_y}} f(i, y)$ which is a subset of $f(n, y)$ where $n \in Z_y$. Therefore $x \in St(y, \gamma_n)$. This implies that $y \in f(n, x)$ and $n \in Z_x$.

Now we prove the converse. Let

$$\gamma_n = \{\{x, y\} \mid y \in g(n, x) \vee x \in g(n, y)\}.$$

Each γ_n is a collection of subsets of X which has two elements. For the sequence $<\gamma_n>$ it can be shown that $\{St(x, \gamma_n)\}$ is a local basis of x. Thus the collection $(\gamma_1, \gamma_2 \cdots\cdots)$ is a qs-development for X.

The above lemma motivates us to formulate the concept of qs-stratifiable spaces by a slight modification of the neccessary condition of lemma 1.5. We state the formal definition as follows:

DEFINITION 2.3. A topological space (x, τ) is said to be *qs-stratifiable* if there exists a mapping $f : N \times X \to \tau$ such that

(1) $Z_x = \{n \mid f(n, x) \neq \phi\}$ is infinite.

(2) x belongs to the intersection of $f(n, x)$ for all n in Z_x.

(3) If x belongs to $f(n, x_n)$ for every n in Z_x, then x belongs to the closure of $\{x_1, x_2, \cdots\cdots\}$ and

(4) n is an element of Z_x whenever $x \in f(n, y)$.

If we set $f(n, x) = Int(g(n, x))$, where $g(n, x)$ is that of lemma 2.2, then cleary $f(n, x)$ satisfies all the above conditions. Therefore every qs-developable space is qs-stratifiable.

LEMMA 2.4. *A space is qs-stratifiable if and only if there is an open covering $<O_n>$ of X such that*

(1) *for each x there exist infinitely many O_n which contain x and*

(2) *for each open set U there corresponds a sequence of closed sets $<U_n>$ such that*

$$U = \bigcup_n (U_n \cap O_n)$$

and

$$U_n \subset V_n \quad if \quad U \subset V.$$

Proof. Suppose X is qs-stratifiable under the mapping $f : N \times X \to \tau$. Let $O_n = \bigcup_{x \in X} f(n, x)$. Then clearly $\bigcup_n O_n = X$. Furthermore each O_n is characterized by the set Z_x, namely,

$$\bigcup_{x \in X} f(n, x) = \{x \in X \mid x \in f(n, x)\}.$$

For an open set U let

$$U_n = X - \bigcup_{x \in X-U} f(n, x).$$

Then $<U_n>$ is a sequence of closed sets. It is not difficult to verify that $U_n = \bigcup (U_n \cap O_n)$.

The remaining part of the theorem is an easy consequence of the fact that $\bigcup_{x \in X-U} f(n, x)$

$\supset \bigcup_{x \in X-V} f(n, x)$ if $U \subset V$.

To prove the converse we set $f(n, x) = (X - (X - x)_n) \cap O_n$.

Then $f : N \times X \to \tau$ and satisfies (1) to (4) of definition 2.3.

THEOREM 2.5. *A space is semi-stratifiable if and only if it is qs-stratifiable and perfect. (For the definition of a semi-stratifiable space see* [6].)

Proof. Suppose X is qs-stratifiable and perfect. Let U be an open set. Then by lemma 2.4 there is a sequence of closed sets $\langle U_n \rangle$ and an open covering $\langle O_n \rangle$ of X such that $U = \bigcup_n (U_n \cap O_n)$. Since X is perfect,

$$U = \bigcup_n (U_n \cap [\bigcup_i F_{ni}]) = \bigcup_{n,i} U_{ni}$$

where each F_{ni} is closed and $U_{ni} = U_n \cap F_{ni}$.

Let U and V be open sets. There correspond two sequences of closed sets $\langle U_n \rangle$, $\langle V_n \rangle$ respectively such that $U_n \subset V_n$ for each n. It follows that $U_{ni} \subset V_{ni}$ since $(U_n \cap F_{ni}) \subset (V_n \cap F_{ni})$. Hence X is semi-stratifiable. The converse is evident by Theorem. 1.2. of [6].

If a space X is first countable, there is a mapping $h : N \times X \to P(X)$ such that $\{h(n, x) \mid n \in N\}$ is a decreasing local basis. Moreover if the space X is qs-stratifiable by the mapping f, then $\{g(n, x) \mid n \in N\}$ is clearly a local basis of X where

$$g(n, x) = f(n, x) \cap h(n, x).$$

The mapping g also satisfies all conditions of definition 2.3. Thus we know that a first countable qs-stratifiable space is qs-developable.

3. M_i-spaces ($i=1, 2, 3$) vs. qs-developable spaces.

Let γ be a collection of subsets of a space. For every subclass γ' of γ if

$$\mathrm{cl}(\bigcup_{c \in \gamma'} C) = \bigcup_{c \in \gamma'} \mathrm{cl}(C),$$

then γ is said to be *closure preserving*. A space is *closure preserving qs-developable* if each γ_n is closure preserving where $\gamma = (\gamma_1, \gamma_2, \cdots\cdots)$ is a qs-development for X.

A regular space X is said to be an *M_1-space* if X has a σ-closure preserving basis. An *M_2-space* is a regular space which has a σ-closure preserving quasi-basis [5].

DEFINITION 3.1. If γ and δ are collections of subsets of X, we say that γ is *cushioned* in δ if there exists a mapping $D : \gamma \to \delta$ such that

$$\mathrm{cl}(\bigcup_{c \in \gamma'} C) \subset \bigcup_{c \in \gamma'} D(C)$$

for every subclass γ' of γ.

A collection of ordered pairs of sets P is called a *pair basis* if

$$P = \{\mathbf{P} = (P_1, P_2) \mid P_i \subset X\}$$

such that

(1) $P_1 \subset P_2$ and P_1 is open, and

(2) for every x and for every neighborhood U of x there exists a $\mathbf{P}$ in P such that

$$x \in P_1 \subset P_2 \subset U.$$

A T_1 space X is said to be an *M_3-space* if X has a σ-cushioned pair basis [5].

It is well known that M_1-space $\rightarrow$ M_2-space $\rightarrow$ M_3-space. For the qs-developable spaces with closure preserving property we have following theorem.

THEOREM 3.2. *A regular and closure preserving qs-developable space is an M_2-space.*

Proof. Let $\gamma=(\gamma_1, \gamma_2 \cdots\cdots)$ be a qs-development for X. Since each γ_n is closure preserving $B_n=\{St(x, \gamma_n) \mid x \in X\}$ is also closure preserving.

THEOREM 3.3. *A regular space X has a closure preserving semi-development if and only if X has a closure preserving qs-development.*

Proof. The neccessity is trivial. Let $\gamma=(\gamma_1, \gamma_2, \cdots\cdots)$ be a closure preserving qs-development for X. Then X is an M_2-space by theorem 3.2. Since X is also an M_3-space, $X-\gamma_n^*=\bigcap_k U_{nk}$ where each U_{nk} is open. Let $\zeta_{nk}=\gamma_n \cup \{U_{nk}\}$. Then for each n and each k ζ_{nk} is clearly a covering of X and is closure preserving. We show that $\zeta=\{\zeta_{nk} \mid n=1, 2, \cdots\cdots; k=1, 2, \cdots\cdots\}$ is a semi-development for X. For each x and each n, k St (x, ζ_{nk}) is a neighborhood of x. Let U be an open set containing x. There exists a number n such that $x \in St(x, \gamma_n) \subset U$. Since $x \in \gamma_n^*$, there exists a number k such that $x \in U_{nk}$. This implies that $x \in St(x, \gamma_n)=St(x, \zeta_{nk}) \subset U$, and completes the proof.

EXAMPLE 3.4. There is a regular closure preserving semi-developable (hence qs-developable by the Theorem 3.3) space which is not metrizable.

Let R be the real line and Q be the set of rational numbers. We also use the notation $\langle x, y\rangle$ denoting the point $(x, y) \in R \times R$ to distinguish it from (s, t) which is an open interval. For $x \in R$ put

$$L_x=\{\langle x, y\rangle \mid \langle x, y\rangle \in R \times R,\ 0<y\}$$

and

$$X=R \cup \{\cup \{L_x \mid x \in R\}.$$

Now we define a basis for X as follows: For $s, t \in Q$ and $z=\langle x, w\rangle \in L_x, 0<s<w<t$, we put

$$U_{s,}{}^{x}{}_{t}(z)=\{<x, y>\mid s<y<t\}$$

and $\mathbf{A}$ to be the set of all such $U_{s,}{}^{x}{}_{t}(z)$. For $r, s, t \in Q$ and $z \in R$, $s<z<t$ and $r>0$, we put

$$V_{r,s,t}(z)=(s, t) \cup (\cup \{\langle w, y\rangle \mid 0<y<r,\ w \in (s, t)-\{z\}\}),$$

and $\mathbf{B}$ to be the set of all such $V_{r,s,t}(z)$.

Now let $\mathbf{U}=\mathbf{A} \cap \mathbf{B}$. Then it can be easily shown that $\mathbf{U}$ is a σ-closure preserving basis making X to be a nonmetrizable first countable M_1-space. For $z \in R$, $s<z<t$ let

$$W_{r,s,t}(z)=((s,t)-\{z\})\cup\{\langle z,y\rangle \mid 0<y<r\}$$

and

$$U_{s,t}=\{U_{s,}{}^{x}{}_{t}(z) \mid s<w<t,\ x\in R \text{ and } z=\langle x,w\rangle\},$$

$$W_{r,s,t}=\{W_{r,s,t}(z) \mid s<z<t \text{ and } z\in R\}.$$

Then

$$\{U_{s,t} \mid s,t\in Q\}\cup\{W_{r,s,t} \mid r,s,t\in Q\}$$

is a closure preserving qs-development for X.

Alexander introduced a class of cushioned pair semi-developable spaces [2] and proved that a space is metrizable if and only if it is T_0 and has a cushioned pair semi-development. We generalize the concept by defining a cushioned pair qs-development and show that such a space is M_3.

DEFINITION 3.5. A space is *cushioned pair qs-developable* if there exist two qs-development γ, δ for the space such that

(1) each γ_n is cushioned in δ_n and

(2) for each x and each open set U containing x there exists a number n such that

$$x\in St(x,\gamma_n)\subset St(x,\delta_n)\subset U.$$

It is not so difficult to show that if X is cushioned pair qs-developable in the sense of [8], that is, if

$$\{\text{the set of isolated points}\}\subset\gamma_1{}^*\subset\gamma_2{}^*\subset\cdots,$$

then this implies the above definition. For if x is not an isolated point and U be an open set containing x, there is an m such that $x\in \mathrm{St}(x,\gamma_m)\subset U$. Since X is T_1, there must exist an $n>m$ such that $x\in\mathrm{St}(x,\delta_n)\subset U$.
For this n, we have $x\in\mathrm{St}\ (x,\gamma_n)\subset\mathrm{St}\ (x,\delta_n)\subset U$.

From the definition cushioned pair semi-development is a cushioned pair qs-development. A cushiond pair qs-developable T_0-space is regular.

Let γ and δ be collections of subsets of a space. We define that γ is *weakly cushioned* in δ if there exists a mapping $D:\gamma\to\delta$ such that

(1) $C\subset D(C)$ for each C in γ and

(2) for each subclass $\gamma'\subset r$

$$\mathrm{cl}(\bigcup_{c\in\gamma'} C)\bigcup_{c\in\gamma'}\mathrm{cl}(D(C)).$$

This is a slight generalization of the definition 3.1.

A space is defined to be *weakly cushioned pair qs-developable* if there exist two qs-development γ and δ such that

(1) each γ_n is weakly cushioned in δ_n and

(2) for each x and each open set U containing x there is a number n such that

$$x \in \mathrm{St}(x, \gamma_n) \subset \mathrm{St}(x, \delta_n) \subset U.$$

It is clear that if X is cushioned pair qs-developable, then it is weakly cushioned pair qs-developable. If a space X is regular and has a closure preserving qs-development $\gamma = (\gamma_1, \gamma_2, \cdots\cdots)$, then it has a weakly cushioned pair qs-development since γ is weakly cushioned in itself and is closure preserving.

THEOREM 3.6. *If a regular space X has a weakly cushioned pair qs-development, then it is an M_3-space.*

Proof. Let γ is weakly cushioned in δ under the mapping D and P_n be a collection of ordered pairs $\mathbf{P} = (P_1, P_2)$ such that $P_1 = \mathrm{Int}\ (\mathrm{St}(x, \gamma_n))$, $P_2 = \bigcup \mathrm{cl}(D(C))$ where C is a member of γ_n containing x. Then P_n is cushioned. It is easy to show that $\bigcup P_n$ is a pair basis. This completes the proof.

From the above theorem we have following corollary which is a generalization of the theorem 2, 3 of [8].

COROLLARY 3.7. *A regular cushioned pair qs-developable space is Nagata (a first countable stratifiable) space.*

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