

ON WAGNER'S GENERALIZED BERWALD SPACE

BY MASAO HASHIGUCHI

In his paper [8] V. Wagner has generalized the notion of a Berwald space, and has obtained an interesting invariant characteristic of a two-dimensional generalized Berwald space. A *Berwald space* means here an affinely connected Finsler space defined by L. Berwald [1, 2] as the space in which his connection coefficients depend on position alone. If we obey the Cartan connection [4], such a space is also the one in which Cartan's connection coefficients $\Gamma^*{}^i{}_j{}_k$ depend on position alone, and is characterized by the well-known condition $C_{ijk;l}=0$. V. Wagner has called a space as a *generalized Berwald space* if there is possible to introduce a generalized Cartan connection, with torsion $(*\Gamma^*{}^i{}_j{}_k - *\Gamma^*{}^i{}_k{}_j \neq 0)$, in such a way that the connection coefficients $*\Gamma^*{}^i{}_j{}_k$ depend on position alone. And in a two-dimensional case he has shown, as the main theorem, that a space is a generalized Berwald space if and only if $\partial A/\partial \theta$ is a function of A , where A and θ are the main scalar and the Landsberg angle respectively (Berwald [2]). Wagner's work is thought to be of great significance in the sense that it offers a good model of Finsler spaces.

The purpose of the present paper is to clarify a meaning of the generalized Cartan connection used in Wagner [8] by some geometrical axioms, and to characterize Wagner's generalized Berwald spaces of general dimensions. In Sections 2 and 3 we shall introduce the notion of metrical connections with deflection and torsion, and define generalized Berwald spaces in a broader sense than Wagner's. In the last section it will be shown that the generalized Cartan connection is a semi-symmetric metrical connection without deflection (Theorem 5), and that a space is Wagner's generalized Berwald space if and only if there exists a generalized Cartan connection satisfying the condition $C_{ijk;l}=0$ similar to the one for a Berwald space (Theorem 7).

Throughout the present paper we shall use the terminologies and notations described in Matsumoto's monograph [7]. It should be remarked that the treated Finsler connections are different from the ones familiar to us. So, for convenience' sake we shall devote Section 1 to sketching the materials necessary for our discussions from his Finsler theory.

The author is indebted to Prof. Dr. M. Matsumoto for drawing his attention to this problem, and wishes to express here his sincere gratitude for the invaluable suggestions and encouragement.

1. Preliminaries.

1.1. Given a differentiable manifold M of dimension n , we denote by $L(M)(M, \pi, GL(n, R))$ the bundle of linear frames and by $T(M)(M, \tau, V, GL(n, R))$ the tangent bundle where the standard fibre V is a vector space of dimension n with a fixed base $\{e_a\}$. The induced bundle $\tau^{-1}L(M) = \{(y, z) \in T(M) \times L(M) \mid \tau(y) = \pi(z)\}$ is called the *Finsler bundle* of M and denoted by $F(M)(T(M), \pi_1, GL(n, R))$. The projection π_1 is the mapping:

$F(M) \rightarrow T(M) | (y, z) \rightarrow y$. The Lie algebra of the structural group $GL(n, R)$ of $L(M)$ and $F(M)$ is denoted by $L(n, R)$ and the canonical base by $\{L_a^b\}$.

Since a point u of $F(M)$ is a pair of a tangent vector y and a linear frame $z = (z_a)$ at a point x of a base manifold M , a coordinate system (x^i) in M induces a coordinate system (x^i, y^i, z_a^i) in $F(M)$ by $y = y^i(\partial/\partial x^i)_x$ and $z_a = z_a^i(\partial/\partial x^i)_x$, which we shall call a *canonical coordinate system* of $F(M)$. The vector y is also called a *supporting element*.

A *Finsler connection* of M is by the first definition of M. Matsumoto a pair (Γ, N) of a connection Γ in the Finsler bundle $F(M)$ and a non-linear connection N in the tangent bundle $T(M)$.

1.2. Given a Finsler connection (Γ, N) , let $l_u(u \in F(M))$ and $l_y(y \in T(M))$ be the respective lifts with respect to Γ and N . In terms of a canonical coordinate system (x^i, y^i, z_a^i) , they are expressed by

$$(1.1) \quad l_u(\partial/\partial x^k)_y = (\partial/\partial x^k)_u - z_b^j \Gamma_j^i{}^k (\partial/\partial z_b^i)_u,$$

$$(1.2) \quad l_u(\partial/\partial y^k)_y = (\partial/\partial y^k)_u - z_b^j C_j^i{}^k (\partial/\partial z_b^i)_u$$

and

$$(1.3) \quad l_y(\partial/\partial x^k)_x = (\partial/\partial x^k)_y - N^i{}_k (\partial/\partial y^i)_y.$$

The $\Gamma_j^i{}^k$, $C_j^i{}^k$ are called the *coefficients of Γ* and the $N^i{}_k$ the *coefficients of N* .

As another lift to $T(M)$ there exists the *vertical lift* l_y^v expressed by

$$(1.4) \quad l_y^v(\partial/\partial x^k)_x = (\partial/\partial y^k)_y.$$

For each $v \in V$ the *h-* and *v-* *basic fields* $B^h(v)$ and $B^v(v)$ are defined by assigning to $u = (y, z)$

$$(1.5) \quad B^h(v)_u := l_u l_y(zv)$$

and

$$(1.6) \quad B^v(v)_u := l_u l_y^v(zv)$$

respectively, where $zv = z_a^i v^a (\partial/\partial x^i)_x$ for $z = z_a^i$, $x = \pi(z)$, $v = v^a e_a$.

Let K be a Finsler tensor field. The *h-* and *v-* *covariant derivatives* of K are defined by $\Delta^h K(v) := B^h(v)K$ and $\Delta^v K(v) := B^v(v)K$ respectively. In terms of a canonical coordinate system, the components of $\Delta^h K$ and $\Delta^v K$ are denoted by $K_j^i{}^k$ and $K_j^i|_k$ respectively, if K is assumed, for instance, to be of type $(1, 1)$, i. e.,

$$(1.7) \quad K = z'^a z_b^j K_j^i e_a \otimes e^b,$$

where $(z'^a) := (z_a^i)^{-1}$, and they are expressed as follows:

$$(1.8) \quad K_j^i{}^k = \delta K_j^i / \partial x^k + K_j^m F_m^i{}^k - K_m^i F_j^m{}^k,$$

$$(1.9) \quad K_j^i|_k = \delta K_j^i / \partial y^k + K_j^m C_m^i{}^k - K_m^i C_j^m{}^k,$$

where $\delta/\partial x^k := \partial/\partial x^k - N^m{}_k (\partial/\partial y^m)$, and

$$(1.10) \quad F_j^i{}^k := \Gamma_j^i{}^k - C_j^i{}^m N^m{}_k,$$

which are the coefficients of the so-called *subordinate V-connection* of (Γ, N) .

1.3. In $F(M)$ the *fundamental vector field* $Z(A)$ is defined for each $A \in L(n, R)$. The vector at $u = (y, z)$ is expressed by

$$(1.11) \quad Z(A)_u = A_b^a z_a^i (\partial / \partial z_b^i)_u,$$

where $A = A_b^a L_a^b$. If a Finsler connection (Γ, N) is given, this field and the basic fields span the tangent space of $F(M)$ at each point. Thus, we have the following structural equations:

$$(1.12) \quad [B^h(1), B^h(2)] = B^h(T(1, 2)) + B^v(R^1(1, 2)) + Z(R^2(1, 2)),$$

$$(1.13) \quad [B^h(1), B^v(2)] = B^h(C(1, 2)) + B^v(P^1(1, 2)) + Z(P^2(1, 2)),$$

$$(1.14) \quad [B^v(1), B^v(2)] = B^v(S^1(1, 2)) + Z(S^2(1, 2)),$$

where we put for brevity $i := v_i \in V$ ($i=1, 2$), and from which we have five kinds of torsion tensor fields T, C, R^1, P^1, S^1 and three kinds of curvature tensor fields R^2, P^2, S^2 . They are called the $(h)h$ -, $(h)hv$ -, $(v)h$ -, $(v)hv$ - and $(v)v$ -*torsion tensor fields* and the h -, hv - and v -*curvature tensor fields* respectively.

In the following discussions we shall use T, P^1, S^1 and P^2, S^2 , whose respective components are as follows:

$$(1.15) \quad T_{jk}^i = \mathfrak{S}_{jk}\{F_{jk}^i\},$$

$$(1.16) \quad P_{jk}^i = \partial N^i_{jk} / \partial y^k - F_{jk}^i,$$

$$(1.17) \quad S_{jk}^i = \mathfrak{S}_{jk}\{C_{jk}^i\}$$

and

$$(1.18) \quad P_{jkl}^i = \partial F_{jk}^i / \partial y^l - C_{j^i l, k} + C_{j^i m} P_{kl}^m,$$

$$(1.19) \quad S_{jkl}^i = \mathfrak{S}_{kl}\{\partial C_{jk}^i / \partial y^l + C_{j^i m} C_{kl}^m\},$$

where $\mathfrak{S}_{jk}\{\dots\}$ denotes, for instance, $\mathfrak{S}_{jk}\{F_{jk}^i\} = F_{jk}^i - F_{kj}^i$.

In the typically used Finsler connections, T and S^1 vanish. In the present paper, however, we shall treat Finsler connections with non-vanishing T .

1.4. Given a Finsler connection (Γ, N) , we have the *associated non-linear connection* N' with the subordinate V -connection Γ_V of (Γ, N) . The pair (Γ, N') is also a Finsler connection. We shall denote by putting a prime the quantities with respect to (Γ, N') . Between the h -basic fields $B^h(\mathfrak{v})$ and $B'^h(\mathfrak{v})$ there exists the relation

$$(1.20) \quad B^h(\mathfrak{v}) = B'^h(\mathfrak{v}) + B^v(D(\mathfrak{v})),$$

where the tensor field D is called the *deflection tensor field*, whose components are expressed by

$$(1.21) \quad D^i_k = y^j F_{jk}^i - N'^i_k.$$

Since the coefficients of N' become $N'^i_k = y^j F_{jk}^i$, the deflection tensor field expresses the difference of N' and N .

Let C be a differentiable curve in M and \tilde{C} be a differentiable curve in $T(M)$ mapped on the C by the projection τ . Tangent vectors $X(t)$ along C are called *parallel along C with respect to \tilde{C}* , if the equations

$$(1.22) \quad \dot{X}^i + \Gamma_{j^i k}^i(x, y) X^j \dot{x}^k + C_{j^i k}^i(x, y) X^j \dot{y}^k = 0$$

are satisfied, where C is expressed by $x^i(t)$ and \tilde{C} by $x^i(t)$, $y^i(t)$, and a dot means d/dt . If we take in particular \tilde{C} to be a lift C^* with respect to the non-linear connection N , the supporting element y satisfies

$$(1.23) \quad \dot{y}^i + N^i_k(x, y) y^k = 0,$$

and so (1.22) may be written in the form

$$(1.24) \quad \dot{X}^i + F_{j^i k}^i(x, y) X^j \dot{x}^k = 0.$$

It is easily seen from (1.21), (1.23) that the supporting element y is parallel with respect to C^* , i. e.,

$$(1.25) \quad \dot{y}^i + F_{j^i k}^i(x, y) y^j \dot{x}^k = 0,$$

if and only if the deflection tensor field vanishes.

Thus, in the typically used Finsler connections, a non-linear connection is chosen such that the deflection tensor field D vanishes. In the present paper, however, we shall also treat Finsler connections with non-vanishing D .

1.5. Now, we shall treat Finsler spaces. Let $L(x, y)$ be the metrical function and G the metric tensor field defined by $g_{ij} := \partial^2(\frac{1}{2}L^2)/\partial y^i \partial y^j$. A Finsler connection in a Finsler space is called *metrical* if the length $(g_{ij}(x, y) X^i X^j)^{1/2}$ of a vector X remains unchanged under parallel displacements along any curve C with respect to any \tilde{C} . Such a connection is characterized by the conditions

$$(1.26) \quad \Gamma_{j^i k}^i + \Gamma_{h^i k}^i = \partial g_{jh} / \partial x^k,$$

$$(1.27) \quad C_{j^i k}^i + C_{h^i k}^i = \partial g_{jh} / \partial y^k,$$

where $\Gamma_{j^i k}^i := g_{ih} \Gamma_{j^i k}^h$, $C_{j^i k}^i := g_{ih} C_{j^i k}^h$, and which are also equivalent to

$$(1.28) \quad g_{ij|k} = 0, \quad g_{ij|_k} = 0,$$

and so each covariant differentiation commutes with the raising and lowering of indices.

If we fix a point x of a base manifold M , the $g_{ij}(x, y)$ give a Riemannian metric on a tangent space at x , from which the Riemannian connection is defined. The Christoffel symbols of the first and second kinds become

$$(1.29) \quad C_{j^i k}^i = \frac{1}{2} \partial g_{jh} / \partial y^k, \quad C_{j^i k}^i = g^{ih} C_{j^i k}^h,$$

which are uniquely determined from the metrical condition (1.27) and the condition of symmetry

$$(1.30) \quad C_{j^i k}^i = C_{k^i j}^i.$$

The condition (1.30) means that the $(v)v$ -torsion tensor field S^1 vanishes. It gets on a fair treat in the cases of metrical connections satisfying $S^1=0$. In the following we shall confine ourselves to such connections. Since the connection coefficients $C_j^i k$ are always given by (1.29), we shall omit the comments for $C_j^i k$ in the statements for Finsler connections. The coefficients $C_j^i k$ and C_{jhk} are symmetric and it holds the so-called C_1 -condition

$$(1.31) \quad y^j C_j^i k = 0, \quad y^j C_{jhk} = 0.$$

And the coefficients (1.19) of S^2 become

$$(1.32) \quad S_j^i k l = \mathfrak{S}_{kl} \{C_j^m l C_m^i k\}.$$

1.6. Since the used Finsler connections are general ones, we have to notice that some formulas have the styles different from the ones familiar to us. For later use we shall show

LEMMA 1. *If a Finsler connection satisfies the C_1 -condition, it holds*

$$(1.33) \quad y^i |_{,j} = D^j y^i, \quad y^i |_{,j} = \delta_j^i,$$

$$(1.34) \quad P_0^i k l = P^i k l + D^i_m C_k^m l + D^i k |_{,l},$$

where δ_j^i are Kronecker's deltas and the subscript 0 means the contraction for the supporting element y .

Proof. The former follows from the definitions of the covariant differentiations and the latter from the contraction of (1.18) for y^h .

LEMMA 2. *If a Finsler connection in a Finsler space is metrical and the connection coefficients C_{jhk} are symmetric, it holds for the components $P_{ijkl} := g_{jh} P_i^h k l$ of the hv-curvature tensor field P^2*

$$(1.35) \quad P_{ijkl} = -P_{jikl},$$

$$(1.36) \quad P_{ijkl} = \mathfrak{S}_{ij} \{C_{jkl|i} - C_{jkm} P^m_{il}\} + A_{ijkl},$$

where putting $T_{jhk} := g_{ih} T_j^i k$

$$(1.37) \quad A_{ijkl} := \frac{1}{2} [(T_{ijk} - T_{jik} + T_{ikj}) |_{,l} \\ + (T_{ijm} - T_{jim} + T_{jmi}) C_k^m l + \mathfrak{S}_{ij} \{(T_{kim} + T_{ikm} + T_{kmi}) C_j^m l\}].$$

Proof. If we apply (1.13) to the metric tensor field G , we have one of the Ricci identities

$$(1.38) \quad g_{ij} |_{,l} k - g_{ij} |_{,k} l = g_{ij} |_{,h} C_k^h l + g_{ij} |_{,h} P^h k l + g_{hj} P_i^h k l + g_{ih} P_j^h k l.$$

The skew-symmetry (1.35) follows from (1.28). If we take the B^h -part of a Jacobi identity

$$(1.39) \quad [B^h(1), [B^h(2), B^v(3)]] + [B^h(2), [B^v(3), B^h(1)]] \\ + [B^v(3), [B^h(1), B^h(2)]] = 0,$$

where $i : = v_i \in V$ ($i=1, 2, 3$), we obtain one of the Bianchi identities

$$(1.40) \quad \mathfrak{E}_{ik} \{C_{jkl} - P_{ijl} + T_{ijm} C_k^m + C_{ijm} P^m_{kl}\} + T_{ijk|l} - T_{imk} C_j^m = 0.$$

The (1.36) follows from the so-called Christoffel process with respect to i , k and j .

2. Metrical connections with deflection and torsion.

2.1. As a famous Finsler connection in a Finsler space there exists the Cartan connection, whose coefficients are given by

$$(2.1) \quad \Gamma_{jkk} = \gamma_{jkk} + \mathfrak{E}_{jh} \{C_{jkm} G^m_h\},$$

$$(2.2) \quad N^i_k = G^i_k,$$

if we put

$$(2.3) \quad \gamma_{jkk} := \frac{1}{2} (\partial g_{jh} / \partial x^k + \partial g_{hk} / \partial x^j - \partial g_{jk} / \partial x^h), \quad \gamma_j^i{}_k := g^{ih} \gamma_{jkk},$$

$$(2.4) \quad G^i := \frac{1}{2} \gamma_j^i{}_k y^j y^k, \quad G^i{}_k := \partial G^i / \partial y^k.$$

M. Matsumoto [6] has proposed the following elegant axioms that determine the Cartan connection.

(C1) *The connection is metrical.*

(C2) *The deflection tensor field D vanishes identically.*

(C3) *The (h)h-torsion tensor field T vanishes identically.*

(C4) *The (v)v-torsion tensor field S^1 vanishes identically.*

The geometrical meanings of the above axioms have been already explained in the previous section. In order to obtain new metrical connections satisfying $S^1=0$, we shall try to replace the axioms (C2), (C3) by some weaker conditions.

2.2. For a Finsler connection (Γ, N) , the axiom (C2) means that the non-linear connection N is just the associated one N' , and the axiom (C3) states that a skew-symmetric Finsler (1,2)-tensor field T vanishes. So, we shall first give any non-linear connection and any skew-symmetric Finsler (1,2)-tensor field. We have

PROPOSITION 1. *Given a non-linear connection N and a skew-symmetric Finsler (1,2)-tensor field T in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the axioms (C1), (C4) and following two.*

(C2') *The non-linear connection is the given N .*

(C3') *The (h)h-torsion tensor field is the given T .*

Let $N^i{}_k$ and $T_j^i{}_k$ be the coefficients of N and the components of T respectively. The coefficients of Γ are given by

$$(2.5) \quad \Gamma_{jkk} = r_{jkk} + \mathfrak{E}_{jh} \{C_{jkm} N^m_h\} + A_{jkk},$$

where putting $T_{jkk} := g_{ih} T_j^i{}_k$

$$(2.6) \quad A_{jkk} = \frac{1}{2} (T_{jkk} - T_{hjk} + T_{jkh}).$$

Proof. We may put

$$(2.7) \quad \Gamma_{j\dot{h}k} = \gamma_{j\dot{h}k} + \mathfrak{S}_{jh} \{C_{jkm} N^m_h\} + A_{j\dot{h}k}$$

for some Finsler (0, 3)-tensor field $A_{j\dot{h}k}$. Due to (1.26) and (1.15) the axioms (C2') and (C3') are expressed by

$$(2.8) \quad A_{j\dot{h}k} + A_{\dot{h}jk} = 0$$

and

$$(2.9) \quad A_{j\dot{h}k} - A_{k\dot{h}j} = T_{j\dot{h}k}$$

respectively. These are uniquely solved and we have (2.6).

For the above connection the deflection tensor field D is expressed by

$$(2.10) \quad D^i_k = G^i_k + 2C^i_{km} G^m - C^i_{km} N^m_0 - N^i_k + A_0^i_k,$$

where $A_j^i_k := g^{ih} A_{j\dot{h}k}$. If we solve (2.10) for N^i_k , we have

$$(2.11) \quad N^i_k = G^i_k - C^i_{km} (A_0^m_0 - D^m_0) + (A_0^i_k - D^i_k).$$

Hence we have obtained

PROPOSITION 2. *Given a Finsler (1, 1)-tensor field D and a skew-symmetric Finsler (1, 2)-tensor field T in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the axioms (C1), (C3'), (C4) and (C2'') The deflection tensor field is the given D .*

Let D^i_k be the components of D . The coefficients of (Γ, N) are given by (2.5) and

$$(2.12) \quad N^i_k = G^i_k - C^i_{km} B^m_0 + B^i_k,$$

where

$$(2.13) \quad B^i_k = A_0^i_k - D^i_k.$$

2.3. The above proposition includes two special cases. If the axiom (C3') remains to be (C3), we have the connection treated by the author [5]. On the other hand, if we impose the axiom (C2) instead of (C2''), the B^i_k in (2.13) become $A_0^i_k$, and we have

PROPOSITION 3. *Given a skew-symmetric Finsler (1, 2)-tensor field T in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the axioms (C1), (C2), (C3') and (C4).*

The coefficients of (Γ, N) are given by (2.5) and

$$(2.14) \quad N^i_k = G^i_k - C^i_{km} A_0^m_0 + A_0^i_k.$$

If we assume that $B^i_k = 0$, which is equivalent to (2.2) owing to (2.12), then we have an example of a Finsler connection with deflection and torsion.

PROPOSITION 4. *Given a skew-symmetric Finsler (1, 2)-tensor field T in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the axioms (C1), (C3'), (C4) and*

(C2'') The non-linear connection N is the one given by E. Cartan. The coefficients of Γ are given by

$$(2.15) \quad \Gamma_{jhk} = \gamma_{jhk} + \mathfrak{S}_{jk} \{C_{jkm} G^m_k\} + A_{jhk}.$$

The deflection tensor field D is expressed by

$$(2.16) \quad D^i_k = A_0^i_k.$$

For simplicity we shall define as follows.

DEFINITION 1. A Finsler connection given in the above any proposition is called a *metrical connection with torsion* if the $(h)h$ -torsion tensor field does not vanish.

3. Generalized Berwald spaces.

3.1. A *Berwald space* is a Finsler space in which the coefficients $F_{j^i k}$ of the Cartan connection depend on position alone, i. e.,

$$(3.1) \quad \partial F_{j^i k} / \partial y^l = 0,$$

where

$$(3.2) \quad F_{j^i k} = \gamma_{j^i k} + g^{ih} C_{jkm} G^m_k - C_{j^i m} G^m_k - C_{k^i m} G^m_j,$$

which are derived from (1.10), (2.1), and which E. Cartan has denoted by $\Gamma^*_{j^i k}$ and V. Wagner by $\{j^i_k\}$. We have generalized the Cartan connection and have obtained metrical Finsler connections with torsion. So, we shall define as follows.

DEFINITION 2. A Finsler space is called a *generalized Berwald space* if there is possible to introduce a metrical Finsler connection with torsion in such a way the connection coefficients $F_{j^i k}$ depend on position alone.

3.2. We shall find the conditions that a Finsler space becomes a generalized Berwald space.

THEOREM 1. A metrical Finsler connection with torsion satisfies the condition (3.1) if and only if it holds

$$(3.3) \quad C_{ijl} D^i_k = 0,$$

$$(3.4) \quad C_{ijk|l} = C_{ij|k} D^m_l,$$

$$(3.5) \quad A_{ijl} = 0,$$

where A_{ijl} are given in (1.37), and are also expressed by A_{ijk} as

$$(3.6) \quad A_{ijl} = A_{ijk}|_l + (A_{ijm} + A_{mij} - A_{mji}) C_k^m_l + \mathfrak{S}_{ij} \{A_{kmi} C_j^m_l\}.$$

Proof. The condition (3.1) holds if and only if the formula (1.18) is reduced to

$$(3.7) \quad P_{ijl} = -C_{ijl|k} + C_{ijm} P^m_{kl}.$$

where the left- and right-hand members are skew-symmetric and symmetric with respect to the indices i and j respectively. Hence, (3.1) is equivalent to

$$(3.8) \quad P_{ij\mu} = 0,$$

$$(3.9) \quad C_{ijk'l} = C_{ijm} P^m_{kl}.$$

By the contraction (3.9) for y^i we have (3.3) from (1.31), (1.33). If we eliminate P^m_{kl} from (3.9) using (1.34), (3.8), we have (3.4). Then the (3.5) follows immediately from (1.36).

Conversely, let us assume (3.3), (3.4) and (3.5). If we write the formula of P^j_{kl} from (1.36), (3.5), and contract for y^i , then we have from (1.31), (1.33), (1.34) and (3.3)

$$(3.10) \quad P^j_{kl} + D^j_m C^m_{kl} + D^j_k |_{l} = C^j_{l10} - C^j_{m} P^m_{0l},$$

from which we have by the contraction for y^k

$$(3.11) \quad P^j_{0l} = -D^j_k |_{l} y^k.$$

Substituting (3.11) into (3.10), the right-hand member of (3.10) vanishes owing to (3.3), (3.4), and we have an expression for P^j_{kl} . The (3.9) follows from (3.3), (3.4), and the (3.8) follows from (1.36), (3.5) and (3.9).

3.3. From Theorem 1 we have

THEOREM 2. *A Finsler space is a generalized Berwald space if and only if there exists a Finsler (1,1)-tensor field D^i_k and a skew-symmetric Finsler (1,2)-tensor field $T^j_k \neq 0$ satisfying the conditions (3.3), (3.4) and (3.5), where the connection is the one given from D^i_k and T^j_k by Proposition 2.*

If we consider Finsler connections without deflection, we have

THEOREM 3. *A Finsler space is a generalized Berwald space if there exists a skew-symmetric Finsler (1,2)-tensor field $T^j_k \neq 0$ satisfying the conditions (3.5) and*

$$(3.12) \quad C_{ijk'l} = 0,$$

where the connection is the one given from T^j_k by Proposition 3.

Corresponding to Proposition 4, we have

THEOREM 4. *A Finsler space is a generalized Berwald space if there exists a skew-symmetric Finsler (1,2)-tensor field $T^j_k \neq 0$ satisfying the condition (3.5) and following two.*

$$(3.13) \quad C_{ijl} A_0^i = 0,$$

$$(3.14) \quad C_{ijk'l} = C_{ijm} |_{k} A_0^m_{l},$$

where the connection is the one given from T^j_k by Proposition 4.

4. Wanger spaces.

4.1. Finally, we shall treat the following special Finsler connections.

DEFINITION 3. A Finsler connection is called *semi-symmetric* if the $(h)h$ -torsion tensor field T has the form

$$(4.1) \quad T_j^i{}_k = \delta_j^i s_k - \delta_k^i s_j$$

for some covariant vector field s_j .

For a semi-symmetric metrical connection, the (2.6) becomes

$$(4.2) \quad A_{jhk} = g_{jk} s_h - g_{hk} s_j,$$

from which we have

$$(4.3) \quad A_j^i{}_k = g_{jk} s^i - \delta_k^i s_j,$$

$$(4.4) \quad A_0^i{}_k = y_k s^i - \delta_k^i s_0, \quad A_0^i{}_0 = L^2 s^i - y^i s_0,$$

where $s^i := g^{ih} s_h$, and the (3.6) is reduced to

$$(4.5) \quad \Lambda_{ijkl} = \mathfrak{S}_{ij} \{g_{ik} (\partial s_j / \partial y^l)\}.$$

Thus, we have

PROPOSITION 5. *For a semi-symmetric metrical connection, the tensor field Λ_{ijkl} vanishes identically if and only if the vector field s_j depends on position alone.*

In the following we shall assume that the field s_j depends on position alone

4.2. We shall restate Propositions 3 and 4 for the semi-symmetric cases.

THEOREM 5. *Given a covariant vector field $s_j(x) \neq 0$ in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the following axioms.*

- (C1) *The connection is metrical.*
- (C2) *The deflection tensor field D vanishes identically.*
- (C3'') *The connection is semi-symmetric with respect to the given s_j .*
- (C4) *The $(v)v$ -torsion tensor field S^l vanishes identically.*

The coefficients $N^i{}_k$ and $F_j^i{}_k$ are given by

$$(4.6) \quad N^i{}_k = G^i{}_k - L^2 C_k^i s^l + y_k s^i - \delta_k^i s_0,$$

$$(4.7) \quad F_j^i{}_k = \Gamma^* j_k + L^2 (S_j^i{}_{kl} + C_j^i{}_m C_k^m{}_l) s^l + (y^i C_{jkl} - y_j C_k^i{}_l - y_k C_j^i{}_l) s^l + C_j^i{}_k s_0 + g_{jk} s^i - \delta_k^i s_j$$

where $C_j^i{}_k$, $S_j^i{}_{kl}$, $G^i{}_k$ and $\Gamma^* j_k$ are given by (1.29), (1.32), (2.4) and (3.2) respectively.

In the two-dimensional case, the above $F_j^i{}_k$ become the coefficients ${}^* \Gamma_j^i{}_k$ ((8) in Wanger [8]) of his generalized Cartan connection, which is thought as a semi-symmetric metrical

connection without deflection. Hence, we shall call the connections given in Theorem 5 the *Wagner connections*.

Corresponding to Proposition 4 we have

THEOREM 6. *Given a covariant vector field $s_j(x)$ in a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the axioms (C1), (C2'''), (C3'') and (C4).*

The coefficients F_j^i are given by

$$(4.8) \quad F_j^i = \Gamma^*_{j^i} + g_{jk} s^i - \delta_k^i s_j.$$

The deflection tensor field D is expressed by

$$(4.9) \quad D^i_k = A_0^i_k = y_k s^i - \delta_k^i s_0.$$

4.3. We have a typical generalized Berwald space corresponding to Theorem 3.

THEOREM 7. *A Finsler space is a generalized Berwald space if there exists a covariant vector field $s_j(x) \neq 0$ such that the Wagner connection defined by the s_j satisfies the condition $C_{ijk;l} = 0$.*

In the two-dimensional case, if we write out concretely the condition $C_{ijk;l} = 0$ using the results in Theorem 5 by the two-dimensional method (Berwald [3]), then we have the condition (15) in Wagner [8]. Thus, we have obtained another proof for his main theorem. We shall call the generalized Berwald spaces defined by Theorem 7 the *Wagner spaces*. Such a space offers a model of Finsler spaces.

On the other hand, the conditions (3.13) and (4.4) imply $C_{ijk} = 0$ for a non-zero $s_j(x)$. So, corresponding to Theorem 4 we have

THEOREM 8. *There does not exist a non-Riemannian generalized Berwald space in the sense of the connection given by Theorem 6.*

References

- [1] L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25**(1926), 40-73.
- [2] L. Berwald, *Über zweidimensionale allgemeine metrische Räume. I, II*, J. reine angew. Math. **156**(1927), 191-210, 211-222.
- [3] L. Berwald, *On Finsler and Cartan geometries, III. Two-dimensional Finsler spaces with rectilinear extremals*, Ann. of Math. (2) **42**(1941), 84-112.
- [4] E. Cartan, *Les espaces de Finsler*, Actualités **79**, Paris, 1934.
- [5] M. Hashiguchi, *On determinations of Finsler connections by deflection tensor fields*, Rep. Fac. Sci. Kagoshima Univ., (Math. Phys. Chem.) **2**(1969), 29-39.
- [6] M. Matsumoto, *A Finsler connection with many torsions*, Tensor (N. S.) **17**(1966), 217-226.
- [7] M. Matsumoto, *The theory of Finsler connections*, Publ. of the Study Group of Geometry, Vol. 5, Department of Math., Okayama Univ., 1970.
- [8] V. Wagner, *On generalized Berwald spaces*, C. R. (Doklady) Acad. Sci. URSS (N. S.) **39** (1943), 3-5.

Kagoshima University