J. Korean Math. Soc.
Vol. 12, No. 1, 1975

QUASI-SUBORDINATE FUNCTIONS AND COEFFICIENT CONJECTURES

BY SUK-YOUNG LEE

1. Introduction.

Let $f(z)$ and $F(z)$ be two functions which are single-valued and analytic in the disc $D=\{z : |z|<R\}$. The function $f(z)$ is said to be *subordinate* to $F(z)$ in D if there exists an analytic function $w(z)$, bounded and regular in D, $|w(z)|\leqq|z|<R$, for which

$$f(z)=F\{w(z)\} \text{ in } D.$$

In this case we write $f(z)\prec F(z)$ in D.

If $g(z)$ and $G(z)$ are two functions analytic in D for which

$$|g(z)|\leqq|G(z)| \text{ in } D,$$

then $g(z)$ is said to be *majorized* by $G(z)$ in D. The concepts of subordination and majorization often play dual or related roles in many theorems involving analytic functions. In order to attempt a unification of the two concepts we introduce the concept of quasi-subordination.

DEFINITION. Let $f(z)=\sum_0^\infty a_n z^n$, $F(z)=\sum_0^\infty A_n z^n$ be analytic in $|z|<R$. Let $\phi(z)$ be a function analytic and bounded for $|z|<R$, $|\phi(z)|\leqq 1$, such that $f(z)/\phi(z)$ is regular and subordinate to $F(z)$ in $|z|<R$. Then $f(z)$ is said to be *quasi-subordinate* to $F(z)$ in $|z|<R$ relative to $\phi(z)$.

If $f(z)$ is quasi-subordinate to $F(z)$ relative to $\phi(z)$ we shall often say simply that $f(z)$ is quasi-subordinate to $F(z)$ and write $f(z)\prec_q F(z)$ for $|z|<R$. We have

$$f(z)=\phi(z)F\{w(z)\},\quad |w(z)|\leqq|z|<R.$$

Two special cases of quasi-subordination are of particular interest:

(1) If $\phi(z)\equiv 1$, then $f(z)$ is subordinate to $F(z)$ for $|z|<R$.

(2) If $w(z)\equiv z$, then $f(z)$ is majorized by $F(z)$ for $|z|<R$.

Unless otherwise specified we shall assume in what follows that $R=1$ and D is the unit disc $E=\{z : |z|<1\}$.

CONJECTURE. (Robertson, 1968) Let $f(z)=\sum_1^\infty a_n z^n \prec_q F(z)=\sum_1^\infty A_n z^n$ in E where $F(z)$ is univalent in E, then

$$|a_n|\leqq n|A_1|,\quad n=1, 2, 3, 4, \cdots\cdots$$

This conjecture was proved to be true for $n=1$, 2, and 3. It is also true for all n if the

Received by the editors Jan. 20, 1975.

coefficients A_n are all real. It is the purpose of this paper to establish theorem 1 and theorem 2 to support the truth of conjecture in case when $F(z)$ is starlike or spiral-like in E and for all sufficiently large values of $n>N_o(F)$.

2. Preliminary results.

We let

$$f(z)=\sum_1^\infty a_n z^n,\quad F(z)=\sum_1^\infty A_n z^n$$

$$\phi(z)=\sum_0^\infty \beta_n y^n,\quad w(z)=\sum_1^\infty d_n z^n$$

$$|\phi(z)|\leqq 1,\quad |w(z)|\leqq|z|,\quad f(z)=\phi(z)F\{w(z)\},\quad z\in E.$$

LEMMA 1. *If* $f(z)\prec_q F_1(z)$ *and* $F_1(z)\prec_q F_2(z)$, *then* $f(z)\prec_q F_2(z)$ *in* E.

Proof. $f(z)=\phi_1(z)F_1\{w_1(z)\}$
$F(z)=\phi_2(z)F_2\{w_2(z)\}$
where ϕ_1, ϕ_2, w_1, w_2 are appropriate bounded analytic functions in E. It follows that

$$f(z)=\phi_3(z)F_2\{w_3(z)\}$$

where $\phi_3(z)=\phi_1(z)\phi_2\{w_1(z)\}$, $|\phi_3(z)|\leqq 1$
and $w_3(z)=w_2\{w_1(z)\}$, $|w_3(z)|=|w_2\{w_1(z)\}|\leqq|w_1(z)|\leqq|z|<1$ in E.
Hence $f(z)\prec_q F_2(z)$ in E.

LEMMA 2. *If* $f(z)\prec_q F(z)$ *in* E *and* $F(0)=0$, *then*

$$f(z)\prec_q \frac{f(z)}{z}\prec_q\frac{F(z)}{z}\quad in\ E.$$

Proof. Since $f(z)=\phi(z)F\{w(z)\}$, $F(0)=0$, $w(0)=0$, we have $f(0)=0$. Thus $\frac{f(z)}{z}$ and $\frac{F(z)}{z}$ are regular in E,

$$\frac{f(z)}{z}=\left(\frac{\phi(z)w(z)}{z}\right)\frac{F\{w(z)\}}{w(z)}.$$

Since $|\phi(z)w(z)/z|\leqq 1$ in E, we have $\frac{f(z)}{z}\prec_q\frac{F(z)}{z}$.
Since $|f(z)|\leqq\left|\frac{f(z)}{z}\right|$ we have $f(z)\prec_q\frac{f(z)}{z}$. By Lemma 1, it follows that $f(z)\prec_q \frac{F(z)}{z}$ in E.

LEMMA 3. *Let* $f(z)\prec_q F(z)$ *in* E. *Then for any* $\rho>0$ *and* $0<r<1$,

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^\rho d\theta\leqq\int_0^{2\pi}\left|F(re^{i\theta})\right|^\rho d\theta.$$

Proof. Since $\dfrac{f(z)}{\phi(z)} \prec F(z)$ in E, we have [11]

$$\int_0^{2\pi} \left| \frac{f(re^{i\theta})}{\phi(re^{i\theta})} \right|^\rho d\theta \leqq \int_0^{2\pi} \left| F(re^{i\theta}) \right|^\rho d\theta$$

and, since $|\phi(z)| \leqq 1$ in E,

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\rho d\theta \leqq \int_0^{2\pi} \left| \frac{f(re^{i\theta})}{\phi(re^{i\theta})} \right|^\rho d\theta.$$

LEMMA 4. *Let* $f(z) = \sum_0^\infty a_n z^n \prec_q F(z) = \sum_0^\infty A_n z^n$ *in* E. *Then*

$$\sum_{k=0}^n |a_k|^2 \leqq \sum_{k=0}^n |A_k|^2, \quad n = 0, 1, 2, 3, \cdots$$

Proof. Let $s_n(z) = \sum_{k=0}^n a_k z^k$, $S_n(z) = \sum_{k=0}^n A_k z^k$, $R_n(z) = \sum_{k=n+1}^\infty A_k z^k$

Then $s_n(z) + \sum_{k=n+1}^\infty a_k z^k = f(z) = \phi(z) F\{w(z)\} = \phi(z) S_n\{w(z)\} + \phi(z) R_n\{w(z)\}$

$$= \phi(z) S_n\{w(z)\} + \sum_{k=n+1}^\infty d_k z^k$$

where $\sum_{k=n+1}^\infty a_k z^k$ and $\sum_{k=n+1}^\infty d_k z^k$ converge for $|z| < 1$.

By definition, $\phi(z) S_n\{w(z)\} \prec_q S_n(z)$.

Hence by Lemma 3 with $\rho = 2$, $z = re^{i\theta}$, $0 < r < 1$,

$$\int_0^{2\pi} \left| \phi(z) S_n\{w(z)\} \right|^2 d\theta \leqq \int_0^{2\pi} |S_n(z)|^2 d\theta$$

$$\int_0^{2\pi} \left| s_n(z) + \sum_{k=n+1}^\infty (a_k - d_k) z^k \right|^2 d\theta \leqq \int_0^{2\pi} \left| S_n(z) \right|^2 d\theta$$

$$\sum_{k=0}^n |a_k|^2 r^{2k} + \sum_{k=n+1}^\infty |a_k - d_k|^2 r^{2k} \leqq \sum_{k=0}^n |A_k|^2 r^{2k}, \quad 0 \leqq r \leqq 1$$

$$\therefore \sum_{k=0}^n |a_k|^2 \leqq \sum_{k=0}^n |A_k|^2, \quad n = 0, 1, 2, \cdots\cdots$$

LEMMA 5. *Let* $f(z) = \sum_0^\infty a_k z^k \prec_q F(z) = \sum_0^\infty A_k z^k$ *in* E. *Let* $F(0) \neq 0$ *and* $f(z) = \phi(z) F\{w(z)\}$ *in* E. *In some neighborhood of the origin, let*

$$\left[\frac{f(z)}{\phi(z)} \right]^{1/2} = \sum_0^\infty b_k z^k, \quad b_0 \neq 0$$

$$\left[F(z) \right]^{1/2} = \sum_0^\infty B_k z^k, \quad B_0 \neq 0,$$

Then $\sum_{k=0}^n |b_k|^2 \leqq \sum_{k=0}^n |B_k|^2$, $n = 0, 1, 2, 3 \cdots\cdots$

Proof. Let $\sigma_n(z)=\sum_{k=0}^{n}b_kz^k$, $M_n(z)=\sum_{k=0}^{n}B_kz^k$, $P_n(z)=\sum_{k=n+1}^{\infty}B_kz^k$

Then $\sigma_n(z)+\sum_{k=n+1}^{\infty}b_kz^k=\left[\frac{f(z)}{\phi(z)}\right]^{1/2}=[F\{w(z)\}]^{1/2}=M_n\{w(z)\}+P_n\{w(z)\}$

$$=M_n\{w(z)\}+\sum_{k=n+1}^{\infty}e_kz^k \text{ for near } z=0.$$

If $\frac{f(z)}{\phi(z)}$ and $F(z)$ have one or more zeros in E, $[f(z)/\phi(z)]^{1/2}$ and $[f(z)]^{1/2}$ would not be regular throughout E. In this case $\sum_{k=n+1}^{\infty}b_kz^k$ and $\sum_{k=n+1}^{\infty}e_kz^k$, though convergent in some neighborhood of the origin, may not separately converge throughout E. But since $w(z)$, $\sigma_n(z)$, $M_n(z)$ are all regular in E the function

$$M_n\{w(z)\}-\sigma_n(z)=\sum_{k=n+1}^{\infty}(b_k-e_k)z^k$$

is regular for $|z|<1$. Hence $\sum_{k=n+1}^{\infty}(b_k-e_k)z^k$ converges in the full unit disc $|z|<1$. Thus $\sigma_n(z)+\sum_{k=n+1}^{\infty}(b_k-e_k)z^k=M_n\{w(z)\}$ and $M_n\{w(z)\}$ is regular and subordinate to $M_n(z)$ in E.

Therefore, $\sum_{k=0}^{n}|b_k|^2\leqq\sum_{k=0}^{n}|B_k|^2$, $n=0,1,2,\cdots\cdots$

LEMMA 6. *Let* $f(z)=\sum_{k=q}^{\infty}a_kz^k$, $a_q\neq 0$,

be quasi-subordinate to $F(z)=\sum_{0}^{\infty}A_kz^k$ *in* E *and* $F(0)\neq 0$. *In some neighborhood of the origin let* $[F(z)]^{1/2}=\sum_{0}^{\infty}B_kz^k$, $B_0\neq 0$,

$$|a_n|\leqq\sum_{k=0}^{n-q}|B_k|^2,\quad n=q,\ q+1,\ q+2,\cdots\cdots$$

Proof. We use the same notation as in Lemma 5. Since $F(0)\neq 0$ and $f(z)/\phi(z)=F\{w(z)\}$, it follows that $f(z)/\phi(z)$ cannot vanish at $z=0$. Let $r_0=\min\{|a|,1\}$, where a is a zero of $f(z)/\phi(z)$ close to the origin, $|a|<1$. Let $r_0=1$ if $f(z)/\phi(z)$ does not vanish in E. Then for $|z|<r_0$

$$\left[\frac{f(z)}{\phi(z)}\right]^{1/2}=\sum_{0}^{\infty}b_kz^k,\quad b_0\neq 0$$

and

$$f(z)=\phi(z)\left(\sum_{k=0}^{\infty}b_kz^k\right)^2$$

$\phi(z)$ has a zero at the origin if $f(z)$ has one and of the same order $q\geqq 0$. For $n\geqq q$ and $r<r_0$ we have

$$a_n=\frac{1}{2\pi i}\oint_{|z|=r}\frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\oint_{|z|=r}\phi(z)\left(\sum_{k=0}^{\infty}b_kz^k\right)^2\frac{dz}{z^{n+1}}$$

$$=\frac{1}{2\pi i}\int_{|z|=r}\phi(z)\left(\sum_{k=0}^{n-q}b_kz^k\right)^2\frac{dz}{z^{n+1}}$$

Since $\phi(z)\left(\sum_{k=0}^{n-q}b_kz^k\right)^2$ is regular in $|z|<1$ the last integral is independent of r for $0<r<1$. For $z=re^{i\theta}$, $0<r<1$,

$$|a_n|\leqq\frac{1}{2\pi r^n}\int_0^{2\pi}|\phi(z)|\ |\sum_{k=0}^{n-q}b_kz^k|^2d\theta\leqq\frac{1}{2\pi r^{n-q}}\int_0^{2\pi}|\sum_{k=0}^{n-q}b_kz^k|^2d\theta$$

$$=\frac{1}{r^{n-q}}\sum_{k=0}^{n-q}|b_k|^2r^{2k}\leqq\frac{1}{r^{n-q}}\sum_{k=0}^{n-q}|b_k|^2$$

Letting $r\to1$ and using Lemma 5 we have

$$|a_n|\leqq\sum_{k=0}^{n-q}|b_k|^2\leqq\sum_{k=0}^{n-q}|B_k|^2,\quad n=q,\ q+1,\ q+2,\cdots\cdots$$

3. Main theorems and proofs.

THEOREM 1. *Let* $f(z)=\sum_{k=q}^{\infty}a_kz^k$, $a_q\neq0$, $q\geqq1$, *be quasi-subordinate to* $F(z)=\sum_{k=1}^{\infty}A_kz^k$ *in* E. *Let* $F(z)$ *be univalent and spiral-like in* E *so that for some real* α, $|\alpha|<\frac{\pi}{2}$, $R_e\left[e^{i\alpha}\frac{F'(z)}{F(z)}\right]>0$ *in* E ($F(z)$ *starlike when* $\alpha=0$). *Then*

$$|a_n|\leqq(n-q+1)\,|A_1|\leqq n\,|A_1|,\quad n=q,\ q+1,\ q+2,\cdots\cdots$$

Proof. If $G(z)=[F(z^2)]^{1/2}=\sum_{k=1}^{\infty}D_{2k-1}z^{2k-1}$

then $R_e\left[e^{i\alpha}\frac{zG'(z)}{G(z)}\right]=R_e\left[e^{i\alpha}\frac{z^2F'(z^2)}{F(z^2)}\right]>0$ in E.

Hence $G(z)$ is an odd function, spiral-like in E. If $P(z)$ denotes an even functoin, regular and having a positive real part in E with $P(0)=1$, then, letting

$$P(z)=\sum_{k=0}^{\infty}p_{2k}z^{2k},\quad e^{i\alpha}zG'(z)=[P(z)cos\alpha+isin\alpha]G(z)$$

we have $e^{i\alpha}(2n-1)D_{2n-1}=e^{i\alpha}D_{2n-1}+cos\alpha\sum_{k=1}^{n-1}P_{2(n-k)}\cdot D_{2k-1}$

$$(n-1)\,|D_{2n-1}|\leqq\sum_{k=1}^{n-1}|D_{2k-1}|,\quad n=2,3,4,\cdots$$

It follows by induction that $|D_{2n-1}|\leqq|D_1|$, $n\geqq2$. From Lemma 6, we have for $n\geqq q$

$$|a_n|\leqq\sum_{k=0}^{n-q}|D_{2k+1}|^2\leqq(n-q+1)\,|D_1|^2=(n-q+1)\,|A_1|\leqq n\,|A_1|$$

THEOREM 2. *Let*

$$f(z)=\sum_{k=1}^{\infty}a_kz^k\prec_qF(z)=\sum_{k=1}^{\infty}A_kz^k\quad in\ E.$$

Let $F(z)$ *be univalent in* E. *If* $F(z)$ *is not the Koebe function* $A_1 z\ (1-\varepsilon z)^{-2}$, $|\varepsilon|=1$, *then*

$$\lim_{n\to\infty}\left|\frac{a_n}{n}\right|<|A_1|$$

Proof. Since

$$[F(z^2)]^{1/2}=\sum_{k=1}^{\infty}D_{2k-1}z^{2k-1},\ D_1^2=A_1,$$

is univalent in E it follows from a theorem due to Hayman [5] that $\lim_{n\to\infty}|D_{2n-1}|=\alpha$ exists and $\alpha<|D_1|$ except for the Koebe function, $F(z)=A_1 z(1-\varepsilon z)^{-2}$, $|\varepsilon|=1$. In this case $|D_{2n-1}|<|D_1|$ for $n>N_0(F)$ and so

$$\lim_{n\to\infty}\left|\frac{a_n}{n}\right|\leqq\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|D_{2k-1}|^2=\alpha^2<|D_1|^2=|A_1|.$$

When $F(z)$ is the Koebe function, which is starlike in E, then $|a_n|\leqq n|A_1|$ for all positive integers by theorem 1.

THEOREM 3. *Let*

$$f(z)=\sum_1^{\infty}a_n z^n \prec_q F(z)=\sum_1^{\infty}A_n z^n,\ f(z)=\phi(z)F\{w(z)\} \quad in\ E.$$

Let $\phi(0)=\beta_0$, $w'(0)=\alpha_1$. *Let* $\mu(\beta_0,\alpha_1)=|\beta_0|+(1-|\beta_0|^2)|\alpha_1|$.
Then

(1) $|a_1|\leqq|\beta_0\alpha_1||A_1|\leqq|A_1|$

(2) $|a_2|\leqq\mu(\beta_0,\alpha_1)\max\{|A_1|,|A_2|\}\leqq\frac{5}{4}\max\{|A_1|,|A_2|\}$.

The inequalities appearing in (1) *and* (2) *are all sharp.*

Proof. We let

$$\phi(z)=\sum_0^{\infty}\beta_n z^n,\ w(z)=\sum_1^{\infty}\alpha_n z^n,\quad |\phi(z)|\leqq1,\quad |w(z)|\leqq|z|.$$

If $f(z)=\phi(z)F\{w(z)\}$, then

$$\sum_1^{\infty}a_n z^n=\left(\sum_0^{\infty}\beta_k z^k\right)[A_1(\alpha_1 z+\alpha_2 z^2+\cdots)+A_2(\alpha_1 z+\alpha_2 z^2+\cdots)+\cdots]$$

$$\therefore\ a_1=\beta_0\alpha_1 A_1 \text{ and } a_2=(\beta_0\alpha_2+\beta_1\alpha_1)A_1+\beta_0\alpha_1^2 A_2.$$

Since $|\beta_0|\leqq1$, $|\alpha_1|\leqq1$, we have $|a_1|\leqq|A_1|$.
Since $|\alpha_2|\leqq1-|\alpha_1|^2$, $|\beta_1|\leqq1-|\beta_0|^2$ for bounded functions $w(z)$ and $\phi(z)$, we have

$$\begin{aligned}|\beta_0\alpha_2+\beta_1\alpha_1|+|\beta_0\alpha_1^2|&\leqq|\beta_0|(1-|\alpha_1|^2)+|\alpha_1|(1-|\beta_0|^2)+|\beta_0\alpha_1^2|\\&=|\beta_0|+(1-|\beta_0|^2)|\alpha_1|=\mu(\beta_0,\alpha_1)\end{aligned}$$

Hence

$$|a_2|\leqq\mu(\beta_0,\alpha_1)\max\{|A_1|,|A_2|\} \tag{3.1}$$

Inequality (3.1) is sharp. For if $0\leqq\beta_0\leqq1$, $0\leqq\alpha_1\leqq1$ and

$$\phi(z)=\frac{\beta_0+z}{1+\beta_0 z}=\beta_0+(1-\beta_0^2)z+\cdots\cdots$$

$$w(z)=\frac{\alpha_1 z+z^2}{1+\alpha_1 z}=\alpha_1 z+(1-\alpha_1^2)z^2+\cdots\cdots$$

then

$$\beta_0\alpha_2+\beta_1\alpha_1+\beta_0\alpha_1^2=\beta_0+(1-\beta_0^2)\alpha_1=\mu(\beta_0,\alpha_1)$$

$$a_2=[\beta_0(1-\alpha_1^2)+(1-\beta_0^2)\alpha_1]A_1+\beta_0\alpha_1^2A_2$$

If $A_1=A_2$, then $a_2=\mu(\beta_0,\alpha_1)A_1=\mu(\beta_0,\alpha_1)\max\{|A_1|,|A_2|\}$ and so (3.1) is sharp. If either $|\beta_0|=1$ or $|\alpha_1|=(1+|\beta_0)|^{-1}$ then $\mu(\beta_0,\alpha_1)\leqq1$ and in this case inequality (3.1) becomes

$$|a_2|\leqq max\{|A_1|,|A_2|\} \tag{3.2}$$

If $|\alpha_1|\leqq\frac{1}{2}$ the inequality $|\alpha_1|\leqq(1+|\beta_0|)^{-1}$ is satisfied and then (3.2) follows. If $|\alpha_1|>\frac{1}{2}$, then $\max_{\beta_0}\mu(\beta_0,\alpha_1)$ occurs for $|\beta_0|=\frac{1}{2}|\alpha_1|<1$ and has the value $(|\alpha_1|+\frac{1}{4}|\alpha_1|)$ which lies in the interval $(1,5/4)$. When

$$f(z)=\left(\frac{1+2z}{2+z}\right)F(z)\quad\text{in } E,$$

and $A_1=A_2$, then $|a_2|=\frac{5}{4}max\{|A_1|,|A_2|\}$. Therefore the constant $\frac{5}{4}$ is best possible.

Acknowledgement.

The author thanks professor Malcolm S. Robertson for his valuable suggestions and criticisms.

References

[1] L. Bieberbach, *Über die koeffizienten derjenigen potenzreihen, welche eine schlichte abbildung des Einheitskreises vermitteln*, S.B. Preuss, Akad. Wiss. **138** (1916), 940–955.

[2] S. Friedland, *On a conjecture of Robertson*, Master of Science Dissertation, Technion–Israel Institute of Technology, Haifa, 1969.

[3] P.R. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal **4** (1955), 427–465.

[4] G.M. Goluzin, *Geometrische Funktionentheorie*, Deutsche verlag der Wissenschaften, Berlin, 1957.

[5] W.K. Hayman, *Multivalent Functions*, Cambridge University Press, Cambridge, England, 1958.

[6] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103–121.

[7] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, 1952.

[8] R.N. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficient*, Notices Amer. Math. Soc. **16** (1968), 181.

[9] M.S. Robertson, *A remark on the odd schlicht functions*, Bull. Amer. Math. Soc. **42** (1936) 366–370.

[10] M.S. Robertson, *The generalized Bieberbach conjecture for subordinate functions*, Michigan Math. J. **12** (1965) 421–429.

[11] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. (2) **48** (1943) 48–82.

[12] Suk-Young Lee, *Class of regular functions for which $zf'(z)$ is spiral*, Korea Society Math. Educ. J. **12** (1973) 4–11.

Chonnam University