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EXTREME POINTS OF THE SHELL OF A LINEAR RELATION

BY SA-GE LEE

Let A be a bounded linear operator defined on a complex Hilbert space H with inner product $<,>$. Let $W(A)=\{<Ax,x>: \|x\|=1,\ x\in H\}$ be the numerical range of A. For each complex number z, let M_z denote the subset of H, $\{x\in H: <Ax,x>=z\|x\|^2\}$. In [5], M. Embry characterized the extreme points of $W(A)$ in terms of M_z. We obtain an analogy of her result in the setting of the shell $s(A)$ of a linear relation A in H (Theorem). In preparing for the proof of our main result, we also get several lemmas which might be useful in their own right.

The notion of the shell $s(A)$ of a linear relation A in a Hilbert space H was introduced by C. Davis in [1], as a solid in the three dimensional Euclidean space R^3. To get familiar with the tools and terminologies which will be used later, we review first some rudiments of [1], [2].

Let $\bar{C}$ denote the extended complex plane, $C\cup\{\infty\}$, and B the unit ball of R^3. Let ζ, h be a complex number and a real, respectively.

We define a mapping $\theta: B\to\bar{C}$, by sending (ζ,h) to the point $z\in C$ such that (ζ,h) is located on the line passing through the point $(z,0)$ and the north pole $(0,1)$ of B. That is, $\theta(\zeta,h)=\dfrac{\zeta}{1-h}, h\neq1$ and $\theta(0,1)=\infty$.

Let S denote the unit sphere $\{(\zeta,h)\in R^3: |\zeta|^2+h^2=1\}$. The stereographic projection $\tau: \bar{C}\to S$ is defined as follows. $\tau(z)=\left(\dfrac{2z}{1+|z|^2},\ \dfrac{-1+|z|^2}{1+|z|^2}\right)$, $z\in C$ and $\tau(\infty)=(0,1)$. Note that $\theta(\tau(z))=z$, for all $z\in\bar{C}$.

A Möbius transformation $\mu: \bar{C}\to\bar{C}$ is defined by sending $z\in C$ to $\mu(z)=\dfrac{az+b}{cz+d}$, where $ad-bc\neq0$ and $\mu(\infty)=\dfrac{a}{c}$. This leads us to define the Möbius transformation, again denoted by $\mu: S\to S$ by sending $\tau(z)$ to $\tau(\mu(z))$, for $z\in\bar{C}$. If we put $\tau(z)=(\zeta,h)$, $\mu(\tau(z))=(\zeta',h')$, then the coordinates are related by the following matrix equation (p. 77 [1]).

$$(1)\qquad \begin{vmatrix} 1+h' \\ \zeta' \\ \bar{\zeta}' \\ 1-h' \end{vmatrix} = \begin{vmatrix} a\bar{a} & a\bar{b} & b\bar{a} & b\bar{b} \\ a\bar{c} & a\bar{d} & b\bar{c} & b\bar{d} \\ c\bar{a} & c\bar{b} & d\bar{a} & d\bar{b} \\ c\bar{c} & c\bar{d} & d\bar{c} & d\bar{d} \end{vmatrix} \begin{vmatrix} 1+h \\ \zeta \\ \bar{\zeta} \\ 1-h \end{vmatrix}.$$

Now if we apply the above equation (1) to the points (ζ,h), (ζ',h') of the unit ball B, where 1's are replaced by $\sqrt{|\zeta|^2+h^2}$, we still get a mapping, also called the Möbius transformation μ of B onto itself, which sends (ζ,h) to (ζ',h'). In the case $d=\bar{a}$, $c=-\bar{b}$, the Möbius transformation μ is just a typical rigid rotation of the unit ball B.

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Let A be a linear relation in H, that is, a linear subspace of $H\oplus H$. The shell $s(A)$ of A is defined as the set all points

$$\left\{\left(\frac{2<y,x>}{\|x\|^2+\|y\|^2}, \frac{-\|x\|^2+\|y\|^2}{\|x\|^2+\|y\|^2}\right) : (y,x)\in A, (y,x)\neq(0,0)\right\}.$$

(p. 70. Definition 1.1 [1]). If $\dim(H)\geq 3$, then $s(A)$ is a convex subset of the unit ball B (p. 304 Theorem 10.1 [2]). Let $I=\{(y,x)\in A : y=x\}$. The point spectrum $\sigma_p(A)$ of A is defined by $\sigma_p(A)=\{z\in C : (A-zI)\cap(\{(0,0)\}\oplus H)\neq\{(0,0)\}\}$, with ∞ adjoined if $0\in\sigma_p(A^{-1})$, where $A^{-1}=\{(x,y)\in H\oplus H : (y,x)\in A\}$. The approximate point spectrum $\sigma_\pi(A)$ of A is the set $\{z\in C :$ There is $(y_n, x_n)\in A-zI, \|x_n\|=1$ and $\|y_n\|\to 0.\}$ (p. 71 Definitions 2.1–2.5, Proposition 2.1 [1]). Then we have

(2) $S\cap s(A)=\tau(\sigma_p(A))$ (p. 72 Theorem2.2 [1]) and

(3) $S\cap \bar{s}(A)=\tau(\sigma_\pi(A))$ (p. 73 Theorem 2.3 [1]),

where $\bar{s}(A)$ denote the closure of $s(A)$ in R^3. The numerical range $W(A)$ of A is defined as the set $\{<y,x> : \|x\|=1, (y,x)\in A\}$, with ∞ adjoined in the case $\infty\in\sigma_p(A)$ (p. 73 Definition 3.1 [1]). It is easy to see that

(4) $\theta(s(A))=W(A)$ (p. 73 Theorem 3.1 [1])

and

(5) $\mu(W(A))=W(\mu(A))$.

For the μ as above, the Möbius transformation $\mu(A)$ of a subset A of $H\oplus H$ (ie, a relation A in H), is defined by

(6) $\mu(A)=\{(ay+bx, cy+dx) : (y,x)\in A\}$ (p. 77 [1]).

For a linear relation A, we have

(7) $\mu(s(A))=s(\mu(A))$ (p. 78 Theorem 5.1 [1]).

The next lemma was obtained by Embry (pp. 647–648, Lemma 1[5]). We state it here without proof.

LEMMA 1. *Let A be a bounded linear operator on a Hilbert space H. For each complex number λ, denote $M_\lambda=\{x\in H : <Ax,x>=\|x\|^2\}$. Let z be in the interior of a line segment with end points a and b in $W(A)$. Let x, y be vectors in H such that $x\in M_a$, $y\in M_b$ with $\|x\|=\|y\|=1$. Then there exist real numbers s and t in the open interval $(0,1)$ and a complex number α, $|\alpha|=1$ such that $tx+(1-t)\alpha y\in M_z$ and $sx-(1-s)\alpha y\in M_z$. Consequently, $M_a\subset M_z+M_z=A$.*

The above lemma is extended easily to the case of a linear relation A.

LEMMA 2. *Let A be a linear relation in a Hilbert space H, with $\infty\notin\sigma(A)$. For each complex number λ, denote $Y_\lambda=\{(y,x)\in A : <y,x>=\lambda\|x\|^2\}$. Let z be in the interior of a line segment with end points a and b in the numerical range $W(A)$ of A. Then $Y_a\subset Y_z+Y_z=A$.*

Proof. Let $(y_1; x_1)\in Y_a$ *and* $(y_2, x_2)\in Y_b$, $\|x_2\|=1$. We want to show that $(y_1, x_1)\in Y_z+Y_z$. Since $\infty\notin\sigma_p(A)$, we may assume that $x_1\neq 0$. Also, since Y_a, Y_z+Y_z are homogeneous, we still can assume that $\|x_1\|=1$. A simple computation shows that x_1 and x_2 must be linearly independent, by using the fact that $\infty\notin\sigma_p(A)$. We consider the Hilbert space H_1 spanned by x_1, x_2, y_1 and y_2. We then find a linear operator A_1 on H_1 into itself such that $A_1x_i=y_i$, $i=1,2$. By applying the previous Lemma 1, we see easily that $(y_1, x_1)\in Y_z+Y_z$, and that $Y_a\subset Y_z+Y_z$. Now $A=\bigcup\{Y_a : a\in W(A)\}\subset Y_z+Y_z$, ie, $A=Y_z+Y_z$ (cf. p. 648 Proofs of Lemma 1, Theorem 1 (iii), [5]). Q.E.D.

COROLLARY 3. *Let A be a linear relation in H with* $\infty\notin\sigma_p(A)$. *Let* $z\in W(A)$ *and* Y_z *be as in the above lemma. If* Y_z *is a linear subspace of A, then z is an extreme point of* $W(A)$ (cf. p. 647 Theorem 1(i) [5]).

Proof. The proof is similar with that of Theorem 1(i), p. 648 [5] and omitted. Q.E.D.

LEMMA 4. *Let A be a linear relation in a Hilbert space H. Let* μ *be a Möbius transformation of A onto another relation A′ in H, by* $(y,x)\rightarrow(ay+bx,\ cy+dx)$, $ad-bc\neq 0$. *Then the following hold.*

(i) *A′ is also a linear relation and* μ *is a topological linear isomorphism on A onto A′, with respect to the norms* $\|(y,x)\|=\|y\|+\|x\|$, $(y,x)\in A$ *and also for* $(y,x)\in A'$.

(ii) *If A is closed, so is A′.*

Proof. The verifications are elementary and omitted. Q.E.D.

The next lemma is also considered as a natural generalization of theorem 1 (ii), p. 647 and Lemma 2, p. 648 [5]. But our proof appears more translucent in the new setting of the linear relation.

LEMMA 5. *Let A be a linear relation in a Hilbert space H with* $\infty\notin\sigma_p(A)$. *As in Lemma 2, let* $Y_\lambda=\{(y,x)\in A : \langle y,x\rangle=\lambda\|x\|^2\}$ *for a complex number* λ. *Let* $z\in W(A)$ *and L be a supporting line of* $W(A)$ *through z. Then the following hold.*

(i) $A_1=\bigcup\{Y_\lambda : \lambda\in L\cap W(A)\}$ *is a linear subspace of A.*

(ii) *If z is an extreme point of* $W(A)$ *then* Y_z *is a linear subspace of A*

(iii) $A_1=A$ *if and only if* $W(A)\subset L$

(iv) *If A is closed, so are* A_1 *and* Y_z

Proof. (i) Note that $\infty\notin W(A)$, since $\infty\notin\sigma_p(A)$. We can find a suitable affine transformation μ of the plane such that the following is true. $\mu(W(A))$ is contained in the closed left half-plane, with respect to the imaginary axis, $\mu(L)$ is the imaginary axis and $\mu(z)=0$, the origin of the plane. Note that $\mu(W(A))=W(\mu(A))$, by the identity (5). Let $[\mu(A)]$ denote the set of all $(y,x)\in H\oplus H$ such that $(y,u),(v,x)\in\mu(A)$ for some $u,v\in H$. Clearly $\mu(A)\subset[\mu(A)]$. We consider the real valued functional f on $[\mu(A)]$, defined by $f(y,x)=\mathrm{Re}\langle y,x\rangle$, where $(y,x)\in[\mu(A)]$. Note that f is a bilinear form on $[\mu(A)]$ with respect to the real scalar multiplication. Let $\mu(A)_1=\{(y,x)\in\mu(A) : \mathrm{Re}\langle y,x\rangle=0\}=\{(y,x)\in\mu(A) : f(y,x)=0\}$. We claim that $u(A)_1$ is a linear subspace of $\mu(A)$. Let $(y_i, x_i)\in\mu(A)_1$, $i=1,2$. Then $f(y_1+y_2,\ x_1+x_2)=f(y_2,x_1)+f(y_1,x_2)\leqq 0$.

Similarly $f(y_1-y_2,\ x_1-x_2)=-f(y_2,x_1)-f(y_1,x_2)\leqq 0$.

If follows that $f(y_1+y_2,\ x_1+x_2)=0$, proving that $\mu(A)_1$ is linear. But a simple computation shows that $\mu(A_1)=\mu(A)_1$. Therefore A_1 is linear as well, by Lemma 4 (i).

(ii) Let Z be an extreme point of $W(A)$. We consider $\mu(A)_0=\{(y,x)\in\mu(A) : \langle y,x\rangle=0\}$, where μ is as in the proof of (i) above.

Since $\mu(Y_z)=\mu(A)_0$, it only needs to show that $\mu(A)_0$ is a linear subspace of $\mu(A)$. But $\mu(A)_0=\{(y,x)\in\mu(A)_1 : \mathrm{Im}\langle y,x\rangle=0\}$. Since 0 is an extreme point of $\mu(A)$, we see that $\mathrm{Im}\langle y,x\rangle\leqq 0$, for all $(y,x)\in\mu(A)_1$ or $\mathrm{Im}\langle y,x\rangle\geqq 0$, for all $(y,x)\in\mu(A)_1$. Let $[\mu(A)_1]$ be similarly defined as $[\mu(A)]$ above. We consider again a real bilinear form g on $[\mu(A)_1]$ by defining $g(y,x)=\mathrm{Im}\langle y,x\rangle$, for $(y,x)\in[\mu(A)_1]$. By the same procedure as for $\mu(A)_1$ and f above, we can conclude that $\mu(A)_0$ is linear.

(iii) Obvious. (iv) It follows from Lemma 4 (ii). Q.E.D.

The necessity implication of the next proposition was overlooked in [5] even for a bounded operator A.

PROPOSITION 6. *Let A be a linear relation in a Hilbert space H with $\infty\notin\sigma_p(A)$. Let $Y\lambda$ denote as in the above Lemma 5. Define $A_1=\bigcup\{Y\lambda : \lambda\in L\cap W(A)\}$. Then A_1 is linear if and only if L is a supporting line of $W(A)$ through z.*

Proof. We only need to prove the necessity. First observe that every point $\lambda\in L\cap W(A)$ can not be located in the interior of a line segment whose end points a, b are in $W(A)$ and $a\notin L\cap W(A)$. For, if it were, then $Y_a\subset Y\lambda+Y\lambda\subset A_1+A_1=A_1$, by Lemma 2, a contradiction. Q.E.D.

LEMMA 7. *Let A be a linear relation in a Hilbert space H. Let $\zeta(y,x)=\dfrac{2\langle y,x\rangle}{\|x\|^2+\|y\|^2}$, $h(y,x)=\dfrac{-\|x\|^2+\|y\|^2}{\|x\|^2+\|y\|^2}$ and $s(y,x)=(\zeta(y,x),\ h(y,x))\in B$, the unit ball of R^3, for $(y,x)\in A$, $(y,x)\neq(0,0)$. Let β be the uniquely determined number, $0\leqq\beta\leqq\infty$ such that $\sup\{h(y,x) : (y,x)\in A\sim\{(0,0)\}\}=\dfrac{-1+\beta^2}{1+\beta^2}$, that is, β is the norm $\|A\|$ of A (of P.81 Definition 7.1 [1]). Then the following hold.*

(i) Let $h_1=\dfrac{-1+\beta^2}{1+\beta^2}$. Then the set $A_1=\{(y,x)\in A : (-\|x\|^2+\|y\|^2)=h_1(\|x\|^2+\|y\|^2)\}$ is a linear relation.

(ii) If A is closed, so is A_1.

Proof. (i) If $h_1=1$, namely $\beta=\infty$, then the proof is obvious. Let $-1\leqq h<1$. It is immediate to see that $A_1=\{(y,x)\in A : \|y\|=\beta\|x\|\}$, and $\|y\|\leqq\beta\|x\|$ for all $(y,x)\in A$. Now let $(y_i,x_i)\in A_1$, $i=1,2$. By the parallelogram law, $\|y_1+y_2\|^2+\|y_1-y_2\|^2=2\|y_1\|^2+2\|y_2\|^2=2\beta^2(\|x_1\|^2+\|x_2\|^2)=\beta^2(\|x_1+x_2\|^2+\|x_1-x_2\|^2)$. Therefore, $\|y_1+y_2\|^2=\beta^2\|x_1+x_2\|^2+\beta^2\|x_1-x_2\|^2\geqq\beta^2(\|x_1+x_2\|^2)$, since $\|y_1-y_2\|\leqq\beta\|x_1-x_2\|$. But $\|y_1+y_2\|\leqq\beta\|x_1+x_2\|$. It follows that $\|y_1+y_2\|=\beta\|x_1+x_2\|$ and $(y_1,x_1)+(y_2,x_2)\in A_1$.

(ii) Straight-forward. Q.E.D.

Our main theorem is an analogy of Theorem 1 (i) [5] of Embry.

THEOREM. *Let A be a linear relation in a complex Hilbert space H of dimension$\geqq 3$.*

For each $u=(\zeta, h)\in s(A)$, *where* ζ *is a complex number*, h *a real, let* $Y_u=\{(y,x)\in A: 2\langle y,x\rangle=\zeta(\|x\|^2+\|y\|^2)$ *and* $-\|x\|^2+\|y\|^2=h(\|x\|^2+\|y\|^2)\}$. *Suppose that* u *is a boundary point of* $s(A)$ *Then* u *is an extreme point of* $s(A)$ *if and only if* Y_u *is a linear subspace of* H.

Proof. In the case that $u\in S$, the unit ball, the assertion can be proved easily by the identity (2). Now let $u\notin S$. Let L denote a supporting plane of $s(A)$ through u. we draw a straight line from the origin of S to the direction of the open halfspace determined by L, that does not meet with $s(A)$, such that the line is also perpendicular to L. Let v be the interesction of S with this line. Then $v\notin s(A)$. Let μ be a Möbius transformation of B which brings v to the north pole of B, by a rigid rotation. Then $\mu(v)\notin s(\mu(A))$ by (7).

Now let L' denote the plane rotated from L by μ. Clearly L' is a supporting plane of $s(\mu(A))$ at $\mu(u)$ and it is parallel to the complex plane. Let $Y_w=\{(0,0)\}\cup\{(y,x)\in\mu(A): s(y,x)=w\in s(\mu(A))\}$. Let $\mu(A)_1=\{Y_w: w\in L'\cap s(\mu(A))\}$.

By Lemma 7 (i), $\mu(A)_1$ is a linear subspace of $\mu(A)$. Note that $s(\mu(A)_1)=L'\cap s(\mu(A))=\mu(L\cap s(A))$. Let $Y=\bigcup\{Y_w: w\in L\cap s(A)\}$.

We claim that $\mu(Y)=\mu(A)_1$ and

(8) $$\mu(Y_u)=Y_{\mu(u)}$$

We shall only verify (8). Let $(y,x)\in Y_u$, so $s(y,x)=u$ (See Lemma 7 for the notation of $s(y,x)$. Then $\mu(u)=\mu(s(y,x))=s(\mu(y,x))$, by (6). It follows that $\mu(y,x)=Y_{\mu(u)}$.

Now $W(\mu(A)_1)=\theta(s(\mu(A)_1))$, by (4). Note that θ is a one to one correspondence which sends a line segment to a line segment. Let $Y_{\theta(\mu(u))}=\{(y,x)\in\mu(A)_1: \langle y,x\rangle=\theta(\mu(u))\|x\|^2\}$.

It is immediate to see that

(9) $$Y_{\mu(u)}=Y_{\theta(\mu(u))}$$

Now we have the following chain of equivalent statements from (a) to (g).

(a) u is an extreme point of $s(A)$.

(b) $\mu(u)$ is an extreme point of $s(\mu(A))$.

(d) $\mu(u)$ is an extreme point of $s(\mu(A)_1)$

(e) $\theta(\mu(u))$ is an extreme point of $W(\mu(A)_1)$

(f) $Y_{\mu(u)}$ is linear (Corollary 3, Lemma 5 (ii) and the above (9))

(g) Y_u is linear (the identity (8)). Q.E.D.

REMARK. Let u be an arbitrary point of $s(A)$ in the above theorem. Question: If Y_u is linear, must u lie on the boundary of $s(A)$? We conjecture that the answer is positive.

References

[1] C. Davis, *The shell of a Hilbert-space operator*, Acta Scien. Math., **29**(1968), 69-86.

[2] C. Davis, *The shell of a Hilbert-space operator*. II Acta Scien. Math., **31**(1970), 301-318.

[3] C. Davis, *Hole-filling theorems for the shell of an operator*, preprint.

[4] H.G. Eggleston, *Convexity*, Cambridge at the University Press, 1966.
[5] M.R. Embry, *The numerical range of an operator*, Pacific J. Math., **32** (1970), 647–650.
[6] J.G. Stampfli, *Extreme points of the numerical range of a hyponormal operator*, Michigan Math. J., **13** (1966), 87–89.
[7] F.A. Valentine, *Convex sets*, MaGraw-Hill, New York, 1964.

Seoul National University