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NOTE ON THE JACOBSON RADICAL IN ASSOCIATIVE TRIPLE SYSTEMS OF SECOND KIND

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1. Introduction.

Let Φ be a commutative associative ring with an identity. A *triple system over* Φ is defined as a unital Φ-module T with a trilinear composition $(x, y, z) \longrightarrow \langle xyz \rangle$. A triple system T over Φ is called an *associative triple system* (ATS) *of first kind* if the trilinear composition satisfies

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle yzu\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

Associative triple systems of first kind have been called *ternary algebras* or *τ-algebras* by Lister [1]. Loos [2] introduces another kind of associative triple system and calls a triple system M over Φ an *associative triple system* (ATS) *of second kind* if the trilinear composition satisfies

$$\langle\langle xyz\rangle uv\rangle = \langle x\langle uzy\rangle v\rangle = \langle xy\langle zuv\rangle\rangle.$$

The basic example of ATS of first kind is a submodule of an associative algebra which is closed relative to $\langle xyz\rangle = xyz$, while the basic example of ATS of second kind is a submodule of an associative algebra with involution which is colsed under $\langle xyz\rangle = x\bar{y}z$. In fact, associative algebras are the only sources for ATS of first and second kind since Lister [1] proves that any τ-algebra is regarded as a submodule of an associative algebra which is closed under xyz and Loos [2] shows that any ATS of second kind is a submodule of an associative algebra with involution which is closed under $x\bar{y}z$.

The Jacobson radical in a τ-algebra is discussed by Lister [1] and has been characterized by Myung [4] in connection with its imbedding. The Jacobson radical for ATS of second kind has been studied by Loos [2]. The purpose of this note is to establish analogous characterizations in τ-algebras given by Myung [4] for an ATS of second kind.

2. Imbedding.

Throughout M will denote an ATS of second kind over the ground ring Φ. An associative algebra B over Φ with an involution $a \longrightarrow j(a) = \bar{a}$ is called an *imbedding* of M if M is a submodule of B which is closed under the ternary product $x\bar{y}z$. If B is an imbedding of M, the smallest imbedding of M contained in B is the j-subalgebra of B generated by M, which we denote by $I = I_B(M)$. Thus

$$I = M + \bar{M} + (M+\bar{M})^2 + (M+\bar{M})^3 + \cdots\cdots$$
$$= \textstyle\sum_{m\geq 0}(M+\bar{M})^{2m+1} + \sum_{n>0}(M+\bar{M})^{2n}.$$

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If we let $T=T(M,\bar{M})=\sum_{m\geq 0}(M+\bar{M})^{2m+1}$, we readily see that T is a τ-algebra relative to $\langle xyz\rangle=xyz$ in I since $T^3\subseteq T$ and that $I=T+T^2$. Hence I is also an imbedding of the τ-algebra T. In particular, if $M+\bar{M}$ is a τ-algebra; that is, $(M+\bar{M})^3\subseteq M+\bar{M}$, then we see that $T=T(M,\bar{M})=M+\bar{M}$ and $T^2=M\bar{M}+M^2+\bar{M}^2+\bar{M}M$. This is always the case when $M^2=0$ since if $M^2=0$, $(M+\bar{M})^3=M\bar{M}M+\bar{M}M\bar{M}\subseteq M+\bar{M}$. Furthermore, if $M^2=0$ then $I=M+\bar{M}+M\bar{M}+\bar{M}M$.

Henceforth we will deal with imbeddings of the form $I=T(M,\bar{M})+T(M,\bar{M})^2$ since any imbedding of M gives rise to an imbedding of this form.

For any ATS of second kind M, Loos [2] and Meyberg [3] construct a more specific imbedding of M, called the "standard" imbedding, which we summarize here in relation to the present imbedding. For any elements x, y, z in M, we set $l(x,y)z=\langle xyz\rangle$ and $r(x,y)z=\langle zyx\rangle$, and let

$$E=End_\Phi M\oplus(End_\Phi M)^{op}$$

where "op" is to mean the the opposite algebra. Define

$$\lambda(x,y)=(l(x,y),l(y,x)),$$

$$\rho(x,y)=(r(y,x),\ r(x,y)).$$

Let L_0 be the submodule of E spanned by all $\lambda(x,y)$, $x,y\in M$ and R_0 be the submodule of E^{op} spanned by all $\rho(x,y)$, $x,y\in M$. Then it is easy to see that L_0 and R_0 are subalgebras of E and E^{op}, respectively. Setting $\overline{\lambda(x,y)}=\lambda(y,x)$ and $\overline{\rho(x,y)}=\rho(y,x)$ yields involutions on L_0 and R_0. We adjoin identities e_1 and e_2 to L_0 and R_0 to get $L=\Phi e_1+L_0$ and $R=\Phi e_2+R_0$. Let $\bar{M}$ now be an isomorphic copy of M. If $a=(a_1,a_2)\in L$, $b=(b_1,b_2)\in R$, and $x\in M$, we define

$$a\cdot x=a_1x,\quad x\cdot b=b_1x,$$

$$\bar{x}\cdot a=\overline{a_2x},\quad b\cdot\bar{x}=\overline{b_2x}.$$

We now consider the module direct sum

$$A=L\oplus M\oplus\bar{M}\oplus R$$

and write every element of A as

$$\begin{pmatrix} a & x\\ \bar{y} & b\end{pmatrix},\quad a\in L,\ b\in R,\ x\in M,\ \bar{y}\in\bar{M}$$

with the natural identification $a=\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}$, $x=\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix}$. Finally we define a product in A by the rule

$$\begin{pmatrix} a & x\\ \bar{y} & b\end{pmatrix}\begin{pmatrix} a' & x'\\ \bar{y}' & b'\end{pmatrix}=\begin{pmatrix} aa'+\lambda(x,y') & a\cdot x'+x\cdot b'\\ \bar{y}\cdot a'+b\cdot\bar{y}' & \rho(y,x')+bb'\end{pmatrix}.$$

Using the above notations and setting, the following theorem can be proved.

THEOREM 1. (*Loos* [1] *and Meyberg* [3]) *Let M be any ATS of second kind. Then we have*

(i) *$A=L\oplus M\oplus\bar{M}\oplus R$ becomes an associative algebra over Φ with identity $e=\begin{pmatrix} e_1 & 0\\ 0 & e_2\end{pmatrix}$ and* *with involution* $u=\begin{pmatrix} a & x \\ \bar{y} & b \end{pmatrix} \to \bar{u}=\begin{pmatrix} \bar{a} & y \\ \bar{x} & \bar{b} \end{pmatrix}$.

(ii) *If* $x, y, z \in M$, *then* $\langle xyz \rangle = x\bar{y}z$ *and so* A *is an imbedding of* M.

(iii) $I_A(M)=L_0 \oplus M \oplus \bar{M} \oplus R_0$ *and* $I_A(M)$ *is an ideal of* A.

(iv) *The Peirce components of* A *with respect to* e_1 *are* $A_{11}=L$, $A_{10}=M$, $A_{01}=\bar{M}$, $A_{00}=R$.

Loos [2] calls the imbedding $A=L \oplus M \oplus \bar{M} \oplus R$ the *standard imbedding* of M. Theorem 1 also shows that any ATS of second kind possesses an imbedding where $M^2=0$. We now return to imbeddings of M of the form $I=T(M, \bar{M})+T(M, \bar{M})^2$ and assume that $M+\bar{M}$ is a τ-algebra and thus $I=M+\bar{M}+(M+\bar{M})^2$. An imbedding $I=M+\bar{M}+(M+\bar{M})^2$ of M is called *direct* if $I=M \oplus \bar{M} \oplus (M \oplus \bar{M})^2$ is a module direct sum.

Suppose that $I=M \oplus \bar{M} \oplus (M \oplus \bar{M})^2$ is a direct imbedding of M such that $M^2=0$ and $Ma=0$ for $a \in \bar{M}M$ and $bM=0$ for $b \in M\bar{M}$ imply $a=b=0$. Then $(M+\bar{M})^2=M\bar{M}+\bar{M}M=M\bar{M} \oplus \bar{M}M$ since if $a \in M\bar{M} \cap \bar{M}M$ then $a=\bar{b}$ for some $b \in M\bar{M}$ and $M\bar{b}=Ma=0$ since $M^2=0$, and so $\bar{b}=0$ or $a=0$. Therefore, in this case, we have that

$$I=M \oplus M \oplus M\bar{M} \oplus \bar{M}M.$$

Let $A_0=L_0 \oplus M \oplus \bar{M} \oplus R_0$ be the imbedding of M in Theorem 1. Define a mapping f from I onto A_0 as

$$f(x+\bar{y}+\textstyle\sum_i x_i\bar{y}_i+\sum_j \bar{u}_j v_j)=x+\bar{y}+\sum_i \lambda(x_i, y_i)+\sum_j \rho(u_j, v_j).$$

Then f is clearly a module homomorphism of I onto A_0. Now suppose $\sum_i \lambda(x_i, y_i)=0$. Then $0=\sum_i \lambda_i(x_i, y_i) \cdot M=\sum_i l(x_i, y_i)M=\sum_i \langle x_i y_i M \rangle$ and hence $0=\sum_i x_i \bar{y}_i M=(\sum_i x_i \bar{y}_i)M$ by the definition of the product in I. Therefore we have that $\sum_i x_i \bar{y}_i=0$ and similarly $\sum_j \rho(u_j, v_j)=0$ implies $\sum_j \bar{u}_j v_j=0$. This proves that f is injective.

We now adjoin an identity e_1 to the algebra $M\bar{M}$ (a subalgebra of I) to get $\Phi e_1+M\bar{M}$, and extend the product in I to $I'=M \oplus \bar{M} \oplus (e_1+M\bar{M}) \oplus \bar{M}M$ by the rule

$$e_1 x=x, \quad x e_1=0, \quad \bar{x} e_1=\bar{x}, \quad e_1 \bar{x}=0,$$
$$e_1 \bar{M}M=\bar{M}M e_1=0$$

for all $x \in M$. One then easily checks that the mapping $\alpha e_1+a \longrightarrow \alpha e_1+\bar{a}, a \in I$, yields an involution on I' extending the involution on I. We further adjoin an identity 1 to I' to get $I''=\Phi \cdot 1+I'$. Then, setting $e_2=1-e_1$, e_2 plays an identity on $\bar{M}M$ since $e_2 \bar{x}y=\bar{x}y-e_1 \bar{x}y=\bar{x}y$. Furthermore, we see that

$$e_1 I'' e_1=\Phi e_1+M\bar{M}, \quad e_1 I''(1-e_1)=M,$$
$$(1-e_1)I'' e_1=\bar{M}, \quad (1-e_1)I''(1-e_1)=\Phi e_2+\bar{M}M.$$

This is to say that $M, \bar{M}$, $\Phi e_1+M\bar{M}$, $\Phi e_2+\bar{M}M$ are precisely the Peirce components of I'' relative to e_1. Hence, by Theorem 1(iv), the function f can be extended to an algebra isomorphism from I'' onto A, the standard imbedding of M. We have therfore proved

THEOREM 2. *Let* $I=M \oplus \bar{M} \oplus (M \oplus \bar{M})^2$ *be any direct imbedding of an ATS of second kind* M *such that* $M^2=0$ *and* $Ma=0$ *for* $a \in \bar{M}M$ *and* $bM=0$ *for* $b \in M\bar{M}$ *imply* $a=b=0$.

Then $I=M \oplus \bar{M} \oplus M\bar{M} \oplus \bar{M}M$ *and* I *is isomorphic to the imbedding* $L_0 \oplus M \oplus \bar{M} \oplus R_0$ *in*

Theorem 1 *as an algebra. Furthermore, there exists an imbedding B of M such that B is isormophic to the standard imbedding of M and $I_B(M)=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$.*

In view of Theorem 2, we will also call a direct imbedding $I=M\oplus\bar{M}\oplus(M\oplus\bar{M})^2$ of M *standard* if $M^2=0$, and $Ma=0$ for $a\in\bar{M}M$ and $bM=0$ for $b\in M\bar{M}$ imply $a=b=0$. Clearly any direct imbedding of M is also regarded as a direct imbedding of the τ-algebra $M+\bar{M}$ in the sense of Lister [1]. Furthermore, Lister [1] calls a direct imbedding $J=T\oplus T^2$ of a τ-algebra T *standard* if $Ta=aT=0$ for $a\in T^2$ implies $a=0$. One easily checks from the definition that if $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ is a standard imbedding of M, then I is also a standard imbedding of the τ-algebra $T=M\oplus\bar{M}$.

3. Characterization of the radical.

Throughout M will denote an ATS of second kind and any imbedding of M will be of the form $I=T(M,\bar{M})+T(M,\bar{M})^2$ such that $M^2=0$. Thus $I=M+\bar{M}+M\bar{M}+\bar{M}M$ and I is an imbedding of the τ-algebra $M+\bar{M}$. Recall that such an imbedding exists for any ATS of second kind; for example, the standard imbedding. For an element $u\in M$, let $M^{(u)}$ be the *u-homotope* of M; that is, the associative algebra defined by

$$x^{u}y=\langle xuy\rangle.$$

An element $x\in M$ is called *properly quasi-invertible* (p.q.i.) in M if x is quasi-invertible (q.i.) in $M^{(u)}$ for all $u\in M$; that is, for every element $u\in M$ there exists an element $y\in M$ such that $x+y=\langle xuy\rangle=\langle yux\rangle$. For $x,y\in M$, define

$$B(x,y)=Id-1(x,y):z\longrightarrow z-\langle xyz\rangle.$$

A submodule V of M is called a *left ideal* of M if $\langle MMV\rangle\subseteq V$, a *right ideal* if $\langle VMM\rangle\subseteq V$, and a *medial ideal* if $\langle MVM\rangle\subseteq V$. V is called an *ideal* of M if it is left, right, and medial. The *Jacobson radical*, *Rad M*, of M is defined to be the set of all p.q.i. elements in M and shown to be an ideal of M (Loos [2] or Meyberg [3]). A right ideal V of M is called (right) *quasi-regular* (q.r.) if $B(v,x)M=M$ for all $v\in V$ and all $x\in M$, and a q.r. left ideal is similarly defined.

LEMMA 3. *Let $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ be a direct imbedding of M such that $M^2=0$. Then, for elements $x,u\in M$, x is q.i. in $M^{(u)}$ if and only if x is q.i. in $I^{(\bar{u})}$.*

Proof. If x is q.i. in $M^{(u)}$, then x is clearly q.i. in $I^{(\bar{u})}$. Let x be q.i. in $I^{(\bar{u})}$ and y be a quasi-inverse of x in $I^{(\bar{u})}$. Then $y=z+\bar{v}+\sum x_i\bar{y}_i+\sum\bar{u}_jv_j$ for z,v,x_i,y_i,u_j,v_j in M, and from $x+y=x\bar{u}y$ we have

$$x+z+\bar{v}+\sum x_i\bar{y}_i+\sum\bar{u}_jv_j=x\bar{u}z+x\bar{u}\bar{v}+\sum x\bar{u}x_i\bar{y}_i+\sum x\bar{u}\bar{u}_jv_j$$

$$=x\bar{u}z+\sum x\bar{u}x_i\bar{y}_i.$$

Since the imbedding is direct, we get

$$x+z=x\bar{u}z=\langle xuz\rangle,\ \sum x_i\bar{y}_i=\sum x\bar{u}x_i\bar{y}_i,\ \sum\bar{u}_jv_j=0.$$

Similarly we show that $x+z=z\bar{u}x=\langle zux\rangle$ and hence z is a quasi-inverse of x in $M^{(u)}$.

A submodule B of an associative algebra A is called a *strict inner ideal* of A if it is *inner*; i.e., $bAb\subseteq B$ for all $b\in B$, and $b^2\subseteq B$ for all $b\in B$. All one-sided ideals (so ideals) of A are strict inner ideals. If M is an ATS of second kind then all left, right, left-right ideals of M, and the submodules $\langle xMx\rangle$ are inner ideals in M. Let V be an inner ideal of M. Then, for every element $x\in M$, $V^{(x)}$ is a strict inner ideal in the x-homotope $M^{(x)}$ since $\langle v\langle xMx\rangle v\rangle\subseteq\langle vMv\rangle\subseteq V$ for all $v\in V$ and $\langle vxv\rangle$, which is the square of v in $M^{(x)}$, is in V. Now, let B be a strict inner ideal of an associative algebra A and let $b\in B$ be q.i. in A. Then $b+a=ab=ba$ for some $a\in A$ and yet $b^2+ab=bab$, so $ab\in B$ since B is strictly inner and so $a\in B$. Hence b is q.i. in B.

LEMMA 4. *Let* $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ *be a direct imbedding of* M *such that* $M^2=0$. *Then, for* $x, y\in M$, *We have*

(i) *If* $x\bar{y}$ *is q.i. in* I *then* $x\bar{y}$ *is q.i. in* $M\bar{M}$ *too and, in particular,* $x\bar{y}$ *is q.i. in* $x\bar{M}$ *or in* $M\bar{y}$.

(ii) *If* $\bar{x}y$ *is q.i. in* I *then* $\bar{x}y$ *is q.i. in* $\bar{M}M$ *and, in particular,* $\bar{x}y$ *is q.i. in* $\bar{M}y$ *or in* $\bar{x}M$.

Proof. (i) Let $t=u+\bar{v}+a+\bar{b}$ be a quasi-inverse of $x\bar{y}$ in I where $u, v\in M$ and $a, b\in M\bar{M}$. From $x+t=x\bar{y}t$ it follows that $v=b=0$ and similarly from $x+t=tx\bar{y}$ we get $u=0$. Hence $t=a\in M\bar{M}$ and $x\bar{y}$ is q.i. in $M\bar{M}$. Since $x\bar{M}$ is a right ideal of $M\bar{M}$, $x\bar{M}$ is strictly inner in $M\bar{M}$. Hence $x\bar{y}$ is q.i. in $x\bar{M}$. The proof for (ii) is now immediate since quasi-invertibility is invariant under an involution.

LEMMA 5. *If* $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$, *then for* $x, y\in M$, *the following are equivalent.*

(i) x *is q.i. in* $M^{(y)}$;
(i′) y *is q.i. in* $M^{(x)}$;
(ii) x *is q.i. in* $I^{(\bar{y})}$;
(ii′) y *is q.i. in* $I^{(\bar{x})}$;
(iii) $x\bar{y}$ *is q.i. in* I;
(iii′) $y\bar{x}$ *is q.i. in* I;
(iv) $B(x, y)$ *is bijective on* M;
(iv′) $B(y, x)$ *is bijective on* M.

Proof. (i)$\Longleftrightarrow$(ii) is Lemma 3. (ii)$\Longrightarrow$(iii): Let z be a quasi-inverse of x in $I^{(\bar{y})}$. Then $x+z=x\bar{y}z=z\bar{y}x$ and hence $x\bar{y}+z\bar{y}=z\bar{y}x\bar{y}=x\bar{y}z\bar{y}$, showing that $z\bar{y}$ is a quasi-inverse of $x\bar{y}$ in I. (iii)$\Longrightarrow$(iv): Let $L(1-x\bar{y})$ be the left multiplication by $1-x\bar{y}$ in I after adjoining an identity 1 to I. Then M is invariant under $L(1-x\bar{y})$ and the restriction of $L(1-x\bar{y})$ to M is $B(x, y)$. By Lemma 4(i), $x\bar{y}$ is q.i. in $x\bar{M}$ and so $(1-x\bar{y})^{-1}=1-x\bar{u}$ for some $u\in M$. Since $L(1-x\bar{y})^{-1}=L(1-x\bar{u})$ and the restriction of $L(1-x\bar{u})$ to M is $B(x, u)$, we have that $B(x, y)^{-1}=B(x, u)$ and hence $B(x, y)$ is bijective on M. (iv)$\Longrightarrow$(i): Since $B(x, y)$ is surjective, $B(x, y)z=-x$ for some $z\in M$ and so $x+z=\langle xyz\rangle$. On the other hand, $B(x, y)\langle zyx\rangle=\langle zyx\rangle-\langle xy\langle zyx\rangle\rangle=\langle zyx\rangle-\langle\langle xyz\rangle yx\rangle=\langle zyx\rangle-\langle xyx\rangle-\langle zyx\rangle=-\langle xyx\rangle=x+z-\langle xy(x+z)\rangle=B(x, y)(x+z)$.

Since $B(x, y)$ is injective, this implies $x+z=\langle zyx\rangle$ and so x is q.i. in $M^{(y)}$. For (iii)$\Longleftrightarrow$(iii′), we simply observe that $\overline{x\bar{y}}=y\bar{x}$ and quasi-invertibility is invariant under an involution.

We now prove the following characterization of p.q.i. elements in M.

THEOREM 6. *If* $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ *then the following are equivalent.*

(i) $x\in M$ *is p.q.i. in* M;

(ii) $x\bar{M}$ *is a q.r. right ideal in* $M\bar{M}$;

(ii′) $\bar{M}x$ *is a q.r. left ideal in* $\bar{M}M$;

(iii) *the principal right ideal* $\langle xMM\rangle$ *is q.r. in* M;

(iii′) *the principal left ideal* $\langle MMx\rangle$ *is q.r. in* M.

Proof. (i)$\Longleftrightarrow$(ii): x is p.q.i. in M if and only if x is q.i. in $M^{(y)}$ for all $y\in M$ if and only if by Lemma 5, $x\bar{y}$ is q.i. in I for all $y\in M$. But then, by Lemma 4, the latter is equivalent to the fact that $x\bar{y}$ is q.i. in $M\bar{M}$ for all $y\in M$.

(i)$\Longleftrightarrow$(iii): It follows from the definition that $B(\langle xyz\rangle, u)=B(x, \langle uzy\rangle)$ and $B(x,y)$ $B(x,-y)=B(x,\langle yxy\rangle)$.

Now, if x is p.q.i. in M, by Lemma 5 $B(x,\langle yzu\rangle)=B(\langle xuz\rangle, y)$ are bijective on M for all $y,z,u\in M$, and so $\langle xuz\rangle$ is p.q.i. in M for all z, $u\in M$. Since *Rad* M, the set of all p.q.i. elements in M, is an ideal of M, this proves that all elements in $\langle xMM\rangle$ are p.q.i. in M. Hence, by Lemma 5, $B(u,v)$ are surjective on M for all $u\in\langle xMM\rangle$ and all $v\in M$, and so $B(u,v)M=M$, showing that $\langle xMM\rangle$ is q.r. in M.

Conversely, if $\langle xMM\rangle$ is q.r. in M, all $B(\langle xyz\rangle, u)=B(x,\langle uzy\rangle)$ are bijective. Hence all $B(x,t)B(x,-t)=B(x,\langle txt\rangle)=B(x,-t)B(x,t)$ are bijective and so all $B(x,t)$ are too: that is, x is p.q.i. in M. Since (i) is left-right symmetric, we get (i)$\Longleftrightarrow$(ii′)$\Longleftrightarrow$(iii′). This completes the proof.

Since a right ideal V of M is q.r. in M if and only if all elements in V are p.q.i. in M, by virtue of Theorem 6 we have the following analogous result of associative algebras.

COROLLARY 7. *Rad* M *is the unique maximal q.r. ideal of* M *containing all q.r. right ideals and q.r. left ideals in* M.

If $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ and $x\in Rad\ M$, by Theorem 6 $x\bar{M}$ is a q.r. right ideal of $M\bar{M}$ and hence by a well known result for associative algebras $x\bar{M}\subseteq Rad\ M\bar{M}$. Conversely, if $x\bar{M}\subseteq Rad\ M\bar{M}$ then $x\bar{M}$ is a q.r. right ideal of $M\bar{M}$ and so by Theorem 6 again x is p.q.i. in M. Hence we have

COROLLARY 8. *If* $I=M\oplus\bar{M}\oplus M\bar{M}\oplus\bar{M}M$ *then*

$$\begin{aligned} Rad\ M &= \{x\in M \mid x\bar{M}\subseteq Rad\ M\bar{M}\} \\ &= \{x\in M \mid \bar{M}x\subseteq Rad\ \bar{M}M\}. \end{aligned}$$

The following characterization of the radical *Rad* M, in view of Theorem 6, can be shown to be exactly the same as in Myung [4].

THEOREM 9. *For any ATS of second kind* M, *we have*

$$\begin{aligned} Rad\ M &= \{x\in M \mid \text{the principal ideal } \langle xMM\rangle \text{ or } \langle MMx\rangle \text{ is q.r. in } M\} \\ &= \{x\in M \mid Rad\ M^{(x)}=M^{(x)}\} \end{aligned}$$

$$=\bigcap_{a\in M} Rad\ M^{(a)}.$$

The proof of the following theorem is also the same as for a τ-algebra (see Myung [4]).

THEOREM 10. *Let M be an ATS of second kind. Then*

(i) *If V is a left-medial (or medial-right) ideal of M then* $Rad\ V=V\cap Rad\ M$.

(ii) *If V is an inner ideal of M, then* $Rad\ V=\{z\in M\mid\langle aza\rangle\in Rad\ M \text{ for all } a\in V\}$.

(iii) *If e is an idempotent in M, i.e.,* $\langle eee\rangle=e^3=e$,

$$Rad\ \langle eMe\rangle=\langle eMe\rangle\cap Rad\ M.$$

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