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THE RADICAL OF TOPOLOGICAL ABELIAN GROUPS

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Introduction.

We will use some definitions from Wright [14] without further reference to them. This paper is divided into two chapters. Chapter I deals with the radical of a general topological abelian group. Wright has defined in [14] the radical in a topological abelian group as the intersection of what he calls the "residual subgroups". However, his theorem 3.3 [14] suggests a more direct definition of the radical; with this new definition we will review some of Wright's theorems and some further theorems. In theorem 1.3 we will show these two definitions are equivalent.

Particular attention has been paid to the radicals of Cartesian product groups; our results, besides being of interest in themselves, will be useful in connection with locally compact abelian groups, with which we will deal in Chapter II.

In Chapter II we consider locally compact abelian groups. Wright has shown that the radical of a locally compact abelian group is the union of all its compact subgroups. Furthermore, he has shown that the radical of a closed subgroup of a locally compact abelian group is the intersection of the radical of the original group and the subgroup itself. Using character groups and one structure theorem, we shall give alternate proofs of these results, and establish certain relations among and properties of $K(G)$, the component of 0 in G, $T(G)$, the radical of G, $K(G)\cap T(G)$, $T(G)+K(G)$, and corresponding subgroups in the character group $\hat{G}$ of G. These theorems extend and sharpen some of the classical results on locally compact abelian groups, including some proved or reproved in papers of Braconnier [4] and Wright [14]; we also obtain Chu's result [6] in the case of a locally compact abelian group. Finally, some consideration has been given to certain structure questions.

Chapter I. The radical in general topological abelian groups.

The radical $T(G)$ of a topological abelian group G is defined as the complementary set of $C(G)$, the conservative of G in G. The first section is a review of some of the fundamental results in [14] from this different, but equivalent, standpoint.

In Section 2, we prove a main theorem 2.1, which, in cooperation with a structure theorem for locally compact abelian groups, plays an important part in the next chapter. Furthermore, we deal with the radical of an everywhere-dense subgroup of a group, and with the radical of a projective limit. Theorem 2.4 could be used to give a simplified proof of theorem 8.9 of [14].

1. Notation and review of Wright's fundamental results on radicals.

We shall be concerned exclusively with topological abelian groups, written additively.

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Accordingly, the identity element will be denoted by 0. Let G be a topological abelian group, A be any subset. $\bar{A}$ denotes the closure of A, $-A$ denotes the set $\{-a : a \in A\}$. If k is any positive integer, the set kA is defined to be the set $\{a_1+a_2+\cdots+a_k : a_i \in A\}$, and $(-k)A=k(-A)$. N means the set of natural numbers. Finally for the open semigroup generated by an open set W, $\bigcup_{n\in N}(nW)$ will be denoted by $O(W)$.

DEFINITION 1.1. In a topological abelian group G, a subset A of G is said to be *0-proper* if $0 \notin A$.

DEFINITION 1.2. Let G be a topological abelian group. The *conservative* of G is the set of elements x in G such that for some open neighborhood U_x of x, $O(U_x)$ is 0-proper: or equivalently the set of all those $x \in G$ which are contained in some 0-proper open subsemigroup of G. The conservative of G will be denoted by $C(G)$.

Suppose G is a topological abelian group containing a 0-proper open semigroup. Then Zorn's lemma can be applied to yield a 0-proper open semigroup with maximal property, that is, a maximal 0-proper open semigroup. We note that if M is a maximal 0-proper open semigroup, then so is $-M$.

DEFINITION 1.3. Let G be a topological abelian group. The *radical* $T(G)$ of G is the complement of $C(G)$ in G.

LEMMA 1.1. *The radical $T(G)$ of G is the set of all elements x such that any neighborhood U_x of x, $O(U_x)$ contains 0 in G, or equivalently $x \in T(G)$ if and only if every open semigroup of G which contains x must also contain 0 in G.*

LEMMA 1.2. *$T(G)$ is a closed subgroup of G.*

Proof. Suppose $x, y \in T(G)$. Let W be any neighborhood of $x-y$ and let U, V be neighborhoods of x and of y respectively such that $U-V \subset W$. Since $x, y \in T(G)$, for some $n \in N$, $0 \in nU$, $0 \in nV$. Hence, $0 \in nU-nV=n(U-V) \subset nW$.

DEFINITION 1.4. Let G be a topological abelian group. If $G=T(G)$, G is said to be a *radical group*, while if $T(G)=0$, G is said to be a *radical-free group*.

DEFINITION 1.5. Let A be any non-void subset of a topological abelian group G. We define $S(A)=\{x \in G : x+A \subset A\}$. $S(A)$ is non-void, since $0 \in S(A)$.

DEFINITION 1.6. A subset B of a topological abelian group G will be called a *residual set* if there exists a maximal 0-proper open semigroup M of G such that $B=S(M)\cap S(-M)$.

By [14] Theorems 2.2 and 3.1, any residual set is a closed subgroup.

THEOREM 1.3. *The radical $T(G)$ of a topological abelian group G is the intersection of all its residual subgroups.*

Proof. A residual subgroup B belonging to M is $B=S(M)\cap S(-M)$, $C(G)=\bigcup_{M}(M$

$\cup(-M))$. By Theorem 3.3 [14], $B=S(M)\cap S(-M)=(M\cup(-M))'$. Hence $T(G)=(C(G))'=(\bigcup_M(M\cup(-M)))'=\bigcap_M(M\cup(-M))'=\bigcap_M(S(M)\cap S(-M))$.

LEMMA 1.4. *A topological abelian group G is a radical group if and only if every open semigroup in G contains an open subgroup of G.*

Proof. If H is an open semigroup, $H\cap(-H)$ is an open subgroup.

LEMMA 1.5. *Let H be a closed subgroup of a topological abelian group G. Then the radical $T(G/H)$ is the set $\{\tilde{x}; O(U_x)\cap H\neq\phi$, for every neighborhood U_x of x, where $\tilde{x}$ is the coset to which x belongs$\}$.*

COROLLARY 1.5.1. *The conservative $C(G/H)$ of G/H is the set $\{\tilde{x}; O(U_x)\cap H=\phi$ for some neighborhood U_x of $x\}$.*

LEMMA 1.6. ([14], Theorem 4.7.) *If G is a topological abelian group with radical $T(G)$, and if H is a closed subgroup of G such that $H\subset T(G)$, then the radical of G/H is $T(G)/H$. In particular, $G/T(G)$ is radical-free.*

Proof. We first remark that the following statements are equivalent: (1) $x\in T(G)$, (2) for every neighborhood U_x of x, $O(U_x)\cap T(G)\neq\phi$, (3) $0\in O(U_x)$. The lemma is obvious from these.

LEMMA 1.7. ([14], Theorem 4.8) *Let G be a topological abelian group with radical $T(G)$. Let H be any closed subgroup of G such that G/H is radical-free. Then $T(G)\subset H$.*

Proof. By Lemma 1.5, $T(G/H)=\{\tilde{x}: 0(U_x)\cap H\neq\phi\}$. Since G/H is radical free, $T(G/H)=\{0\}$. This shows $\{x: O(U_x)\cap H\neq\phi\}=H$. Obviously for any subset K of G, if $H\supset K$, then $\{x: O(U_x)\cap H\neq\phi\}\supset\{x: O(U_x)\cap K\neq\phi\}$. In particular, putting $K=\{0\}$, $H\supset T(G)$.

The next theorem is a slight generalization of theorem 3.3.

THEOREM 1.8. *Let G be a topological abelian group and H be a subgroup of G. If G/H and H are both radical groups, then so is G.*

Proof. For any $x\in G$, and any neighborhood U_x of x, $O(U_x)\cap H\neq\phi$, since G/H is a radical group. On the other hand H is a radical group. Hence, $0\in O(U_x)$.

LEMMA 1.9. *Let G be a topological abelian group with radical $T(G)$. If $T(G)$ is open then it is a radical subgroup of G.*

Before proceeding to the next section, we will provide some examples of radicals of topological abelian groups.

EXAMPLE 1. *The reals R.* $(-\infty, 0)$, $(0, +\infty)$ are obviously maximal 0-proper open semigroups in R. Let $M=(0,+\infty)$; then $(-\infty,0)=-M$, $S(M)=[0,\infty)$, $S(-M)=(-\infty,0]$. Hence $b(M)=S(M)\cap S(-M)=(0)$. Thus, R is radical-free.

EAXMPLE 2. *Discrete groups.* If G is a discrete abelian group, the radical $T(G)$ of G is the torsion subgroup of G. Choosing $\{x\}$ as U_x, $x\in T(G)$ if and only if the cyclic group (x) generated by x has finite order.

REMARK: Let G have two compatible topologies J_1, J_2 such that $J_1\subset J_2$, and let $C(J_1)$, $C(J_2)$ be the conservatives of $(G_1\ J_1)$, $(G_1\ J_2)$ respectively. Then $C(J_1)\subset C(J_2)$, or $T(J_1)\supset T(J_2)$. Since the radical of a discrete group is the torsion subgroup and the radical of an indiscrete group is the group itself by the following Example 4, $T(G)$ lies between G and its torsion subgroup.

EXAMPLE 3. *Compact groups.* Let G be any compact group, and let S be a semigroup in G. Then S is necessarily a closed subgroup of G under either of the following conditions: (1) S is closed, (2) S is open. More generally, any compact semigroup which satisfies the cancellation laws is a group. This proves that a compact group is a radical group.

EXAMPLE 4. *Indiscrete groups.* Obviously, these are radical groups. Any radical subgroups of a topological abelian group G is included in $T(G)$ the radical of the group. In particular, compact subgroups and indiscrete subgroups are included in the radical of the original group.

Let G be any topological abelian group which is not T_0. Then the closure of the identity, $\{0\}^-$, is an indiscrete subgroup of G. Hence it is included in $T(G)$, and $T(G)/\{0\}^-$ is the radical of $G/\{0\}^-$, which is a T_0 group. Thus we may assume henceforth that all groups are T_0.

EXAMPLE 5. *A locally convex linear topological space E.* Suppose $x\in E$, $x\neq 0$. Then we can choose a convex neighborhood V of x such that $0\notin V$. Now $\bigcup_{n\in N} nV$ is 0-proper: for otherwise, for some positive integer n_0, $0\in n_0V$. Hence, $y_1+y_2+\cdots+y_{n_0}=0$, where $y_i\in V$. Thus $0=\frac{1}{n_0}y_1+\frac{1}{n_0}y_2+\cdots+\frac{1}{n_0}y_{n_0}\in V$, and E is radical free.

2. The radicals of Cartesian product groups and other special groups.

THEOREM 2.1. *Let each G_α $(\alpha\in A)$ be a topological abelian group, and let $G=\prod_{\alpha\in A}G_\alpha$ be the Cartesian product group of the G_α. Then $T(G)=\prod_{\alpha\in A}T(G_\alpha)$, where $T(G_\alpha)$ $(\alpha\in A)$ is the radical of G_α.*

Proof. Let $D=\bigcup_{\alpha\in A}(C(G_\alpha)\times\prod_{\alpha'\neq\alpha}G_{\alpha'})$, where $C(G_\alpha)$ $(\alpha\in A)$ is the conservative of G_α. It is obvious that $D\subset C(G)$. For the converse, let $x\in C(G)$. Then there exists a 0-proper open-semigroup M in G such that $x\in U_{\alpha_1}\times U_{\alpha_2}\times\cdots\times U_{\alpha_n}\times\prod_{\alpha'\neq\alpha_i(i=1,2,\cdots,n)}G_{\alpha'}\subset M$. Let $B=U_{\alpha_1}\times U_{\alpha_2}\times\cdots\times U_{\alpha_n}\times\prod_{\alpha'\neq\alpha_i}G_{\alpha'}$. We remark that for any integer m, $mB=(mU_{\alpha_1})\times(mU_{\alpha_2})\times\cdots\times$

$\prod_{\alpha'\neq\alpha_i} G_{\alpha'}$. Since $B\subset M$, and M is a semigroup, $mB\subset M$ for any $m\in N$. Hence, $O(B)\subset M$, and $0_G\notin mB(m\in N)$, where 0_G is the identity of G. On the other hand, suppose $0_{G\alpha_1}\in m_1U_{\alpha_1},\cdots, 0_{G\alpha_n}\in m_nU_{\alpha_n}$, for some $m_i\in N$ $(i=1,2,\cdots\cdots,n)$. Then $0_G\in m_1m_2\cdots m_nB$. Hence for some i, $1\leq i\leq n$, $O(U_{\alpha_i})$ is a 0-proper open semigroup in G_{α_i}, or $O(B)\subset C(G_{\alpha_i})\times\prod_{\alpha\neq\alpha} G_{\alpha}$.

COROLLARY 2.1.1. *Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a non-void family of topological abelian groups, and $G=\prod_{\alpha\in\Lambda}G_\alpha$. Let G_0 be the local direct product of the G_α's relative to the open subgroups $H_\alpha(\alpha\in\Lambda)$. Then $T(G_0)$ consists of all (x_α), for which $x_\alpha\in T(H_\alpha)$ for all but a finite number of indices α', $x_{\alpha'}\in T(G_{\alpha'})$.*

THEOREM 2.2. *If H is an everywhere dense subgroup in G, then $C(H)=C(G)\cap H$ and $T(H)=T(G)\cap H$.*

Proof. That $C(H)\supset C(G)\cap H$ is obvious. Conversely, let $x\in C(H)$. Then there exists a 0-proper open semigroup M of H such that $x\in M\subset C(H)$. Since M is open in H, there exists a neighborhood U of x in G such that $x\in U\cap H\subset M$. This implies that $O(U\cap H)$ is 0-proper, or $0\notin n(U\cap H)$ for any $n\in N$. Hence $O(U)$ is 0-proper in G.

COROLLARY 2.2.1. *H is a radical subgroup if and only if $\bar{H}$ is a radical subgroup.*

COROLLARY 2.2.2. *Let $G^\omega=\prod^*_{\alpha\in\Lambda}G_\alpha$, the weak Cartesian product of the topological abelian groups $G_\alpha(\alpha\in\Lambda)$. Then $T(G^\omega)=\prod^*_{\alpha\in\Lambda}T(G_\alpha)$.*

Theorem 2.2 provides an interesting example: Let G be a locally compact abelian group, $\bar{G}$ its Bohr compactification, and β the map which maps G continuously isomorphic into $\bar{G}$. Then $\beta(G)$ is dense in $\bar{G}$. Since $\bar{G}$ is a radical group, so is $\beta(G)$. However, G itself is not necessarily a radical group. In particular, $\beta(R)$ is a radical group and R is radical-free.

The next theorem is concerned with an inverse system. Let $\{G_\lambda\}$ be a family of topological abelian groups indexed by a directed set $\Lambda\{\lambda:>\}$, and for any pair $\lambda,\mu\in\Lambda$ such that $\lambda>\mu$, let $f_{\lambda\mu}$ be an open continuous homomorphism of G_λ into G_μ. Let $S=\{G_\lambda: f_{\lambda\mu}\}$ be the inverse system, and $'G$ its projective limit. For each open set U_λ of G_λ, we introduce the set $V_\lambda=\{x\in{}'G,\ x_\lambda\in U_\lambda\}$. Having $\{V_\lambda\}$ as a basis, the topology of $'G$ coincides with the relative topology in G. With the above notations we can state the following theorem.

THEOREM 2.3. *The radical $T('G)$ of $'G$ is the intersection of $\tilde{T}_\lambda$ $(\lambda\in\Lambda)$, where $\tilde{T}_\lambda=\{x\in{}'G; x_\lambda\in T(G_\lambda)\}$; and $C('G)=\bigcup_{\lambda\in\Lambda}\{x\in{}'G; x_\lambda\in C(G)_\lambda\}$.*

Proof. It is obvious that $\bigcup_{\lambda\in\Lambda}\{x\in{}'G; x_\lambda\in T(G_\lambda)\}\subset C('G)$. Let M be any 0-proper open semigroup in $'G$. Let $x\in M$, and choose an open set V_λ such that $x\in V_\lambda\subset M$. Then $x\in O(V_\lambda)=\bigcup_{n\in N}(n\{x\in{}'G; x_\lambda\in U_\lambda\})=\bigcup_{n\in N}\{x\in{}'G; x_\lambda\in nU_\lambda\}=\{x\in{}'G; x_\lambda\in O(U_\lambda)\}\subset M$. Hence, $0_\lambda\notin O(U_\lambda)$, where 0_λ is the identity of G_λ. In other words $U_\lambda\subset C(G_\lambda)$, which implies $x\in V_\lambda\subset\{x\in{}'G; x_\lambda\in C(G_\lambda)\}$.

THEOREM 2.4. *If H is a radical subgroup of a topological abelian group G such that G/H is discrete, then the radical of G is $\pi^{-1}(D)$, where π is the natural homomorphism of G onto G/H and D is the torsion subgroup of G/H.*

Proof. Obvious, since $D=T(G/H)$ and H is open.

We remark that theorem 2.4 is an extension of the statement in Example 2 that the radical of a discrete group is the torsion subgroup of the original group, and that if G/H is a radical group (torsion-free group) then G is a radical group (H is the radical of G).

Chapter II. Locally compact abelian groups.

In this chapter we deal exclusively with locally compact abelian groups. Section 3 is composed entirely of notations. Theorems which will be applied rather frequently through the rest of this paper will be found in [7], [10], [11], and [13]. Section 4 is the main-body of this chapter. The main theorems are theorems 4.2, 4.3, and 4.9. Applying these theorems together with fundamental theorems in locally compact abelian groups, we investigate vertical and horizontal relations between a locally compact abelian group and its character groups, and certain of their subgroups.

Section 5 is related to a result of Chu in the case of locally compact abelian groups, and finally, in Section 6, we consider some particular questions concerning the structure of a locally compact abelian group G, with respect to $K(G)$, $T(G)$, $K(G)\cap T(G)$, and $T(G)+K(G)$.

3. Preliminary notations.

Suppose G is a locally compact abelian group. For $x\in G$, we understand by (x) the group generated by x. For convenience, the component of the identity of G is denoted by $K(G)$.

DEFINITION 3.1. In a locally compact abelian group G, for $x\in G$, if $(x)^-$ is compact, x is called a *compact element* and if (x) is isomorphic (topologically) with the integers, then x is called an *integral element*.

DEFINITION 3.2. A *character* χ of a locally compact abelian group G is a continuous homomorphism $\chi: G\to T$ (the multiplicative group of the unit circle in the complex plane). Let us denote the set of all characters on G by $\hat{G}$.

DEFINITION 3.3. Let $\hat{G}$ be the character group of the group G, and H a subgroup of G. Let us denote by $(H)^0$ the set of all elements $\alpha\in\hat{G}$ for which $\alpha(x)=1$ for every $x\in H$. The set $(H)^0$ is called the annihilator of H in $\hat{G}$, and is a closed subgroup of $\hat{G}$. Let P be a subgroup of $\hat{G}$, and let us denote by $(P)_0$ the set of all elements $x\in G$ for which $\alpha(x)=1$ for every $\alpha\in P$. The set $(P)_0$ is called the annihilator of P in G, and is a closed subgroup of G.

4. The radical of a locally compact abelian group.

Our task is to prove Theorem 4.3, Wright's main theorem ([14], Theorem 8.10). Theorem 4.2 which was originally proved by Pontrjagin [11] and Braconnier [4] plays an important role in obtaining this result.

At first let us denote by C the set of all compact elements in a locally compact abelian group G.

LEMMA 4.1. *If $ny\in C$ for some positive integer n, then $y\in C$. Hence G/T is torsion-free.*

Proof. Suppose for a positive integer n and $y\in G$, $ny\in C$. Since $ny\in C$, $(ny)^{-}$ is compact, $(y)^{-}=\bigcup_{t=0}^{n-1}\{ty+(ny)^{-}\}$ is compact.

THEOREM 4.2. *Let G be a locally compact abelian group and $\hat{G}$ be the character group of G. The annihilator of the component of the identity of G forms the set of compact elements $\hat{C}$ in $\hat{G}$; in other words, $(K(G))^{0}=\hat{C}$.*

Proof. Let us denote by π the canonical homomorphism from G onto $G/K(G)$. Suppose $h\in(K(G))^{0}$. Since h is a character of G, h is continuous from G to T (Unit circle). Therefore, for any neighborhood V of 1 in T, say, $V=\{z\in T:\ z=e^{i\theta},\ -\frac{1}{10}<\theta<\frac{1}{10}\}$, there exists a compact neighborhood U of 0 in G such that $h(U)\subset V$. Since $h\in(K(G))^{0}$, $h(K(G)+U)\subset V$. Since $G/K(G)$ is totally disconnected, and $\pi(U)$ forms a compact neighborhood of the identity in $G/K(G)$, there exists a compact open subgroup H^{*} of $G/K(G)$ such that $H^{*}\subset\pi(U)$. Since $\pi^{-1}(H^{*})\subset\pi^{-1}(\pi(U))=K(G)+U$, $h(\pi^{-1}(H^{*}))\subset h(K(G)+U)\subset V$. Then $h(\pi^{-1}(H^{*}))=(0)$, since, otherwise, $h(\pi^{-1}(H^{*}))\not\subset V$. $\pi^{-1}(H^{*})$ is an open subgroup in G. Hence h belongs to some compact subgroup of $\hat{G}$, which implies $(K(G))^{0}\subset C$. Conversely, let $k\in\hat{C}$. Then $(k)^{-}$ is compact. Let $M=(k)^{-}$. Since $L=(M)_{0}$ is an open subgroup in G, $(M)_{0}\supset K(G)$. Hence $M=((M_{0}))^{0}\subset(K(G))^{0}$, which implies $\hat{C}\subset(K(G))^{0}$.

COROLLARY 4.2.1. $(K(\hat{G}))_{0}=C$, $K(G)=(\hat{C})_{0}$, *and* $K(\hat{G})=(T)^{0}$.

THEOREM 4.3. ([14], Theorem 8.10). *Let G be a locally compact abelian group. Then the following sets in G are identical*: (1) *the radical $T(G)$ of G*, (2) *the maximal radical subgroup M of G*, (3) *C, the set of compact elements x in G.*

Proof. $T(G)\supset M$, since M is a radical subgroup. For $x\in C$, $(x)^{-}\subset M$. Hence, $T(G)\supset M\supset C$. For $C\supset T(G)$, it is sufficient to show G/C is radical-free. By corollary 4.2.1, G/C is isomorphic to the character group of $K(\hat{G})$, and $K(\hat{G})$ is expressed as the cartesian product of R^{n} and H where H is a compact connected group. Hence $G/C\cong R^{n}\times D$, where D is isomorphic to the character group of H, and therefore is a discrete group. D must be a torsion-free group, for otherwise G/C has an element of finite order which contradicts lemma 4.1. Thus G/C is radical-free.

COROLLARY 4.3.1. $T(G)=(K(\hat{G}))_{0}$, $K(\hat{G})=(T(G))^{0}$, $K(G)=(T(\hat{G}))_{0}$, *and* $(K(G))^{0}=T(\hat{G})$.

COROLLARY 4.3.2. *If G is a locally compact abelian group with radical $T(G)$, and if H is a closed subgroup of G, then the radical of H is $H \cap T(G)$.*

COROLLARY 4.3.3. *In a locally compact abelian group G if x is an integral element then there exists a 0-proper open semigroup which contains x.*

THEOREM 4.4. *A locally compact abelian group is a radical group (totally disconnected) if and only if its character group is totally disconnected (a radical group).*

Proof. By corollary 4.3.1, it is immediate.

THOREM 4.5. *A locally compact abelian group is connected (radical-free) if and only if its character group is radical-free (connected).*

Proof. By corollary 4.3.1, it is obvious.

THEOREM 4.6. ([14], Theorem 8.4). *If G is a connected locally compact abelian group, then the radical $T(G)$ is connected and compact.*

Proof. $G \cong R^n \times H$, where H is a connected compact group. Hence $T(G) \cong T(R^n) \times T(H) \cong T(H) = H$.

THEOREM 4.7. ([14], Theorem 8.3). *If G is a connected radical-free locally compact abelian group, then G is topologically isomorphic to a vector group of (unique) finite dimension.*

Proof. Since G is connected, $G \cong R^n \times H$, where H is compact. Since G is radical-free $T(G) = H = (0)$.

THEOREM 4.8. *For a locally compact abelian group G, $T(K(G)) = K(T(G)) = K(G) \cap T(G)$, a compact connected subgroup of G.*

Proof. $K(T(G)) \subset T(K(G))$, since $K(T(G))$ is a radical subgroup by corollary 4.3.2. On the other hand $T(K(G)) \subset K(T(G))$, since $T(K(G))$ is connected by theorem 4.6. Applying corollary 4.3.2 again $T(K(G)) = K(G) \cap T(G)$.

COROLLARY 4.8.1. ([14], Lemma 8.6). *If G is a locally compact radical abelian group, then the component of the identity in G is compact.*

COROLLARY 4.8.2. ([14], Theorem 8.7) *A necessary and sufficient condition that a connected locally compact abelian group be a radical group is that G be compact.*

THEOREM 4.9. *$K(G) + T(G)$ is open.*

Proof. Since $G \cong R^n \oplus G$, $T(G) \subset G'$, and G' includes a compact open subgroup, $T(G)$ is open in G'. Hence $K(G) + T(G) = ((R^n \oplus (K(G) \cap T(G))) + T(G)) = R^n \oplus T(G)$, which is open in G.

COROLLARY 4.9.1. ([14], p. 491) *If G is a totally disconnected locally compact abelian group, then $T(G)$ is open.*

THEOREM 4.10. *A locally compact abelian group G is compact if and only if G is a radical group and $G/K(G)$ is compact.*

Proof. If G is compact G is a radical group and $G/K(G)$ is compact. On the other hand, if G is a radical group, $K(G)=K(T(G))$ is compact by theorem 4.8.

THEOREM 4.11. *A locally compact abelian group G is discrete if and only if $T(G)$ is discrete and G is totally disconnected.*

Proof. If $T(G)$ is discrete and G is totally disconnected, then $K(G)+T(G)=T(G)$ is discrete so that $K(G)+T(G)$ is open and discrete in G.

In fact, theorem 4.11 is a dual of theoerm 4.10.

COROLLARY 4.11.1. ([14], Lemma 8.1) *Let G be a totally disconnected radical-free group. Then G is discrete.*

THEOREM 4.12. ([14], Theorem 8.5) *Let G be a radical-free locally compact abelian group. Then $K(G)$ is topologically isomorphic to a Euclidean vector group R^n, $G/K(G)$ is discrete, and G is topologically isomorphic to $(G/K(G))\times K(G)$.*

Proof. By corollary 4.3.2 and theorem 4.7, $K(G)$ is isomorphic to a Euclidean vector group; in particular, it is a divisible subgroup of G. Hence $K(G)$ is a direct summand (in the algebraic sense) in G. Let us assume $G\cong K(G)+L$ (in the algebraic sense), where L is a complementary summand of $K(G)$. By theorem 4.9, since G is a radicalfree $K(G)=K(G)+T(G)$ is open in G.

Since $K(G)$ is also divisible, $G\cong K(G)\times G/K(G)$.

THEOREM 4.13. *Let ν_k be the canonical mapping from a locally compact abelian group G onto $G/K(G)$. Then $\nu_k(T(G))=T(G/K(G))\cong T(G)/(K(G)\cap T(G))$.*

Proof. That $\nu_k(T(G))\cong(T(G)+K(G))/K(G)$ is obvious.

Now $G\cong R^n\times G'$, $T(G)\subset G'$. Hence for any closed subset H of $T(G)$ saturated with respect to the equivalence relation modulo $T(G)\cap K(G)$, we have $(H+K(G))\cap T(G)=[(R^n\oplus(K(G)\cap T(G)))+H]\cap T(G)=[R^n\oplus((K(G)\cap T(G))+H)]=(R^n\oplus H)\cap T(G)=H$, since $H\cap T(G)=H$ and $T(G)\cap R^n=(0)$. Furthermore, $K(G)+H$ is closed in G and saturated with respect to the equivalence relation mod $K(G)$. Hence $(T(G)+K(G))/K(G)=T(G)/(T(G)\cap K(G))$. On the other hand since $T(G)\supset K(G)\cap T(G)$, by theorem 1.6. $T(G/K(G))\cong T(G'/(K(G)\cap T(G)))=T(G')/(K(G)\cap T(G))=T(G)/(K(G)\cap T(G))$. But the isomorphism $\eta: G'/(K(G)\cap T(G))\longrightarrow G/K(G)$, maps for $t\in G'$, $t+(K(G)\cap T(G))$ to $t+K(G)$. This shows $T(G/K(G))=\nu_k(T(G))$.

COROLLARY 4.13.1. ([11], Theorem 41) *Let G be a locally compact abelian group, $K(G)$ its component of 0. If $G/K(G)$ is compact, then $T(G)$ is compact and $G=K(G)+T(G)=R^n\oplus T(G)$ for some non-negative integer n; the converse is also true.*

Proof. Suppose $G/K(G)$ is compact. Then $T(G/K(G))=G/K(G)$. Hence by the previous theorem $T(G)/(K(G)\cap T(G))$ is compact. By theorem 4.8, $K(G)\cap T(G)$ is compact. Then $T(G)$ is compact. Since $\nu_K(T(G))=t(G/K(G))=G/K(G)$, $G=T(G)+K(G)=R^n\oplus T(G)$ for some non-negative integer n. Conversely, if $G=T(G)+K(G)$ and $T(G)$ is compact, then $T(G)/(T(G)\cap K(G))\cong G/K(G)$ is compact.

THEOREM 4.14. *Let ν_T be the canonical mapping from a locally compact abelian group G onto $G/T(G)$. Then $\nu_T(K(G))=K(G/T(G))\cong K(G)/(K(G)\cap T(G))$.*

Proof. Let us recall first that $K(G)+T(G)$ is open in G, whence $K(G/T(G))=\nu_T(\overline{T(G)+K(G)})$. Hence $\nu_T((\overline{T(G)+K(G)}))=\nu_T(T(G)+K(G))=\nu_T(K(G))$. On the other hand, $\nu_T(K(G))\cong(T(G)+K(G))/T(G)$. The proof that $(T(G)+K(G))/T(G)\cong K(G)/(K(G)\cap T(G))$ is similar to the corresponding part of the proof of the previous theorem.

THEOREM 4.15. *Let G be a locally compact abelian group. Then the followings are equivalent:* (1) *$K(G)$ is compact,* (2) *$K(G)$ is a radical subgroup,* (3) *there exists in G a compact open subgroup,* (4) *$T(G)$ is open,* (5) *$K(G)\subset T(G)$,* (6) *$K(G)/(K(G)\cap T(G))$ is compact,* (7) *$K(G)/(K(G)\cap T(G))=(0)$,* (8) *$K(G/T(G))$ is compact, and* (9) *$K(\hat{G})$ is compact.*

Proof. By the previous theorem $K(G/T(G))\cong K(G)/(K(G)\cap T(G))$. Hence (6) and (8) are equivalent. By corollary 4.3.1, (4) and (9) are equivalent.

(6)$\longrightarrow$(1). Since $K(G)/(K(G)\cap T(G))$ is compact, by theorem 4.8, $K(G)\cap T(G)$ is compact. Then $K(G)$ is compact.

(1)$\longrightarrow$(3). $G/K(G)$ is totally disconnected. Hence $G/K(G)$ contains a compact open subgroup. Furthermore, since $K(G)$ is compact, G itself contains a compact open subgroup.

(3)$\longrightarrow$(4). By hypothesis and theorem 1.10, $H\subset T(G)$ and therefore $T(G)$ is open.

(4)$\longrightarrow$(5). Since $T(G)$ is open, $K(G)\subset T(G)$.

(5)$\longrightarrow$(7). Since $K(G)\subset T(G)$, $K(G)\cap T(G)=K(G)$. This shows $K(G)/(K(G)\cap T(G))=(0)$.

(7)$\longrightarrow$(2). By theorem 4.14, $\nu_T(K(G))=K(G)/(K(G)\cap T(G))$. This shows $\nu_T(K(G))=(0)$. Hence $K(G)\subset T(G)$. Hence by corollary 4.3.2, $K(G)$ is a radical subgroup.

(2)$\longrightarrow$(6). By theorem 4.8, since $K(G)$ is a radical group, $K(G)$ is compact. Hence (6) holds.

In theorem 4.15, (1) and (9) are equivalent. Therefore, from condition (2) through (8) if we replace G by $\hat{G}$, these are also equivalent conditions.

THEOREM 4.16. *Let G be a locally compact abelian group. Then the following are equivalent:* (1) *$K(G)$ is open,* (2) *$T(\hat{G})$ is compact,* (3) *$(T(G)+K(G))/K(G)$ is discrete,* (4) *$T(G)/(K(G)\cap T(G))$ is discrete, and* (5) *$K(G)\cap T(G)$ is open relative to $T(G)$.*

Proof. By corollary 4.3.1, (1) and (2) are equivalent. On the other hand by theorem 4.13, (3), (4), and (5) are equivalent. $T(G)+K(G)$ is open. Hence $(T(G)+K(G))/K(G)$ is discrete if and only if $K(G)$ is open in G.

THEOREM 4.17. *Let G be a locally compact abelian group. Then the following are equivalent:* (1) *$T(G)$ is compact,* (2) *$K(\hat{G})$ is open,* (3) *$\hat{G}/K(\hat{G})$ is discrete, and* (4) *$T(G)/(K(G)\cap T(G))$ is compact.*

Proof. By corollary 4.3.1, (1) and (2) are equivalent. Obviously (2) and (3) are equivalent. By theorem 4.8 $K(G)\cap T(G)$ is compact. Hence (1) and (4) are equivalent.

THEOREM 4.18. *Let G be a locally compact abelian group. Then the following are equivalent:* (1) *$T(G)$ is connected,* (2) *$T(G)\subset K(G)$,* (3) *$G/K(G)$ is radical-free.* (4) *$G/K(G)$ is discrete and torsion-free,* (5) *$K(G)$ is open and for each $x\notin K(G)$, $(x)\cap K(G)=(0)$, and* (6) *$\hat{G}/K(\hat{G})$ is radical-free.*

Proof. By corollary 4.3.1, $T(G)$ is the character group of $\hat{G}/K(\hat{G})$. Thus we have (1) equivalent to (6) by theorem 4.5.

(1)$\longleftrightarrow$(2). If $T(G)$ is connected $T(G)\subset K(G)$. On the other hand if $T(G)\subset K(G)$, $T(G)$ can be considered as the radical of $K(G)$. But by theorem 4.6, the radical of a connected group is connected.

(3)$\longleftrightarrow$(4)$\longleftrightarrow$(5). It is easily seen that (4)$\longleftrightarrow$(5). To show (3)$\longleftrightarrow$(4). Since $G/K(G)$ is totally disconnected, it is discrete if it is radical-free, by corollary 4.11.1. Since a discrete group is radical-free if and only if it is torsion-free, (3)$\longleftrightarrow$(4) holds.

(2)$\longleftrightarrow$(4). Since $T(G)\subset K(G)$, $T(G)+K(G)=K(G)$ is open. Hence $G/K(G)$ is discrete. By theorem 4.13, $T(G/K(G))\cong T(G)/(K(G)\cap T(G))=T(G)/T(G)=(0)$. This shows $G/K(G)$ is radical-free. Hence $G/K(G)$ is discrete and torsion-free. On the other hand since $G/K(G)$ is radical-free, by Theorem 1.7, $T(G)\subset K(G)$.

REMARK. Since (3) and (6) are equivalent, if we replace G by $\hat{G}$ in (1), (2), (4), and (5), the theorem still holds.

THEOREM 4.19. *Let G be a locally compact abelian group. Then the following are equivalent:* (1) *$T(G)$ is discrete,* (2) *$\hat{G}/K(\hat{G})$ is compact,* (3) *$T(\hat{G})$ is compact and $K(\hat{G})+T(\hat{G})=\hat{G}$, and* (4) *$K(G)$ is open and $K(G)\cap T(G)=(0)$.*

Proof. (1)$\longrightarrow$(2). Since $T(G)$ is a character group of $\hat{G}/K(\hat{G})$, it follows from corollary 4.3.1.

(2)$\longrightarrow$(3). It is obvious by corollary 4.13.1.

(3)$\longrightarrow$(4). Since $T(\hat{G})$ is compact, $K(G)$ is open by theorem 4.16. Since $(K(\hat{G})+T(\hat{G}))_0=(K(\hat{G}))_0\cap(T(\hat{G}))_0=T(G)\cap K(G)$ and $K(\hat{G})+T(\hat{G})=\hat{G}$, $T(G)\cap K(G)=(0)$.

(4)$\longrightarrow$(1). If $K(G)$ is open, then $T(G/K(G))$ is discrete. On the other hand $T(G/K(G))=T(G/(T(G)\cap K(G))$ by theorem 4.13. But by assumption $K(G)T(G)=(0)$. This shows $T(G)$ is discrete.

The following, Theorem 4.20, is a generalization of Corollary 4.20.1, which was originally established by Braconnier ([4], p.51).

THEOREM 4.20. *Let G be a connected locally compact abelian group. Then G is a torsion group if and only if $G=(0)$.*

Proof. If G is (0), it is trivially a torsion group. Since the mapping $\eta_n(x) : x \to nx$ is a continuous endomorphism, the kernel K_{η_n} of η_n, $\{x : nx=0,\ x \in G\}$, forms a closed subgroup in G. Since G is a torsion group, $\bigcup_{n \in N}(K_{\eta_n})=G$. Since G is a set of second category, at least one of K_{η_n}, $K_{\eta_{n0}}$, contains some open set of G. Hence $G=K_{\eta_{n0}}$, since G is connected. Therefore for any $\lambda \in \hat{G}$, $n_0\lambda(G)=\lambda(n_0G)=\lambda(0)=0$. Hence $\hat{G}$ is a torsion group. Hence $K(G)=(0)$ by theorem 4.4.

COROLLLARY 4.20.1. (See [4], p. 51). *If G is a locally compact torsion group, then* $K(G)=K(\hat{G})=(0)$.

Proof. $K(G)$ is a connected locally compact torsion group. Hence by the previous theorem $K(G)=(0)$. Then by theorem 4.4, $K(\hat{G})=(0)$.

5. Chu's result in case of locally compact abelian groups.

In [6] Chu has shown a locally compact group which contains a syndetic cyclic subgroup, but contains no compact subgroup other than identity, is isomorphic to either the reals or the integers. In the case of a locally compact abelian group such a group is a radical-free group containing a syndetic cyclic group. We establish this for the abelian case by the following comparatively simple method.

THEOREM 5.1. *If G is a locally compact abelian group, radical-free, and for some $x \in G$, $G/(x)$ is compact, then G is either isomorphic to R (the reals) or to Z (the integers).*

LEMMA 1. *In theorem* 5.1, *if G is discrete then $G \cong Z$.*

Proof. (See[6]).

LEMMA 2. *If a locally compact abelian group G is a finite dimensional real vector group, then $G \cong R$.*

Proof. Suppose $G=R^n$ for some positive integer n. By a certain linear transformation, it can easily be seen that $G=L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where each L_i is isomorphic with R and $x \in L_1$. On the other hand $((x)\oplus L_2 \oplus \cdots \oplus L_n)/(x)$ is a closed subgroup of the compact group $G/(x)$ and is therefore compact. But $((x)\oplus L_2 \oplus \cdots \oplus L_n)/(x) \cong L_2 \oplus \cdots \oplus L_n$.

Hence $L_2 \oplus \cdots \oplus L_n=(0)$, or $G=L_1$, that is, $G \cong R$.

Proof of theorem 5.1. By theorem 4.12 G is isomorphic with $R^n \oplus H$, where H is a torsion-free discrete subgroup of G.

Case 1. $x \in H$.

Then (x) is a discrete subgroup of G and hence $R^n \oplus (x)$ is a closed subgroup in G, implying that $(R^n \oplus (x))/(x)$ is closed in $G/(x)$. But $(R^n \oplus (x))/(x) \cong R^n$, $(R^n \oplus (x))/(x)$ is compact. Therefore G is a discrete group and hence isomorphic to Z by lemma 1.

Case 2. $x \in R^n$.

$(x)\oplus H$ is a closed subgroup in G and so is $((x)\oplus H)/(x)$ in $G/(x)$. Since $((x)\oplus$

$H)/(x)\cong H$, we must have $H=(0)$. Hence $G\cong R$ by Lemma 2.

Case 3. $x=y+z$, $y\in R^n$, $z\in H$, $y\neq 0$, $z\neq 0$.

Since G is a direct sum of R^n and H, $M=(y,z)$, the group generated by y and z, is expressed as $M=(y)\oplus(z)$. Since (y) is discrete in R^n and so is (z) in H, M is a discrete subgroup in G and therefore closed. Since $M/(x)$ is closed in $G/(x)$, $M/(x)$ is compact. On the other hand the basis in M can be chosen as $\{x, -z\}$ or $M=(x)\oplus(-z)$. Hence $M/(x)\cong(-z)$, so that $(-z)$ is compact. But since $z\in H$, $(-z)$ has to be a finite discrete group. This is a contradiction.

6. Some considerations on particular structure theorems.

In a locally compact abelian group G, we have investigated, to some extent, relations among the following subgroups of G; $K(G)$, $T(G)$, $K(G)\cap T(G)$, and $T(G)+K(G)$.

Along with general structure theorems of locally compact abelian groups, the conjecture naturally arises: Whether some of the following relations are true?

(1) $G\cong K(G)\times G/K(G)$,

(2) $G\cong T(G)\times G/T(G)$,

(3) $G\cong(K(G)\cap T(G))\times G/(K(G)\cap T(G))$, and

(4) $G\cong(T(G)+K(G))\times G/(T(G)+K(G))$.

But the following example shows none of them is true in general.

At first an example that a torsion subgroup of a certain discrete abelian group is not a direct summand of the original group, was given by F. Levi [9]. Here we quote an example from Kaplansky [8].

EXAMPLE. "Let G be the complete direct sum (in algebraic sense) of cyclic groups of orders $p, p^2, p^3, \cdots$ for some prime number p, and let T be its torsion subgroup". Then T is not a direct summand of G. Hence if G is given the discrete topology, (2) fails to be true. This example also shows that the compact character group $\hat{G}$ of G cannot be decomposed as $K(\hat{G})\times\hat{G}/K(\hat{G})$. Moreover, since in this case $\hat{G}=T(\hat{G})$, $K(\hat{G})=K(\hat{G})\cap T(\hat{G})$, and $(K(G))^0\cap((T(G))^0=(K(G)+T(G))^0$, (3) and (4) are false in general. (2) is the dual of (1). Therefore, this one example covers each case. However, it is still interesting to deal with this conjecture.

THEOREM 6.1. *Let G be a locally compact abelian group. Suppose (α), (β), and (γ) are as follows:*

(α) $G\cong G/K(G)\times K(G)$,

(β) $G\cong G/(T(G)+K(G))\times(T(G)+K(G))/K(G)+K(G)/(K(G)\cap T(G)\times (K(G)\cap T(G))$, *or equivalently* $G\cong G/(T(G)+K(G))\times(T(G)+K(G))/K(G)\times K(G)$,

(γ) $(T(G)+K(G))/K(G)$ *is an algebraic direct summand of* $G/K(G)$.

Then $(\alpha)\cup(\gamma)=(\beta)$.

Proof. (*) Let us recall at first that if H is an open subgroup of G, and is an algebraic direct summand of G, then $G\cong H\times G/H$ (see Theorem 4.12).

Suppose (β) is true. Then $G \cong G/(T(G)+K(G)) \times (T(G)+K(G))/K(G) \times K(G)$. Let us denote $A = G/(T(G)+K(G)) \times (T(G)+K(G))/K(G) \times K(G)$ and $B = G/(T(G)+K(G)) \times (T(G)+K(G))/K(G)$. Let η be an isomorphism of A onto G. Then $\eta(K(G)) = K(G)$, since $\eta(K(G))$ is the component of the range group G. Hence $G/K(G) \cong B$.

Suppose f is an isomorphic mapping of B onto $G/K(G)$. Then since $T(G/K(G)) = (T(G)+K(G))/K(G)$, $(T(G)+K(G))/K(G)$ is the (topological) direct summand of $G/K(G)$.

On the other hand, by the previous remark (*) it is obvious that $(\alpha) \cup (\gamma)$ implies (β), since $(T(G)+K(G))/K(G)$ is open in $G/K(G)$ and is an algebraic direct summand of $G/K(G)$.

Let us remark, in the above theorem, $G/(T(G)+K(G))$ is a discrete torsion-free group, $(T(G)+K(G))/K(G)$ is a totally disconnected radical group, $K(G)/(K(G) \cap T(G))$ is isomorphic with R^n for some non-negative integer n, that is, connected and radical-free, and $K(G) \cap T(G)$ is a connected compact (radical) group.

THEOREM 6.2. *If* $K(G)$ *is open, then* $G/K(G) \times K(G) \cong G$.

Proof. $K(G)$ is a divisible subgroup of G and hence an algebraic direct summand of G. Then by the previous remark (*), the theorem holds.

THEOREM 6.3. *If* $T(G)$ *is compact, then* $G \cong T(G) \times G/T(G)$.

Proof. If $T(G)$ is compact, then $K(G)$ is open and the theorem holds by duality.

THEOREM 6.4. *If* (1) $K(G) \supset T(G)$, *or if* (2) $T(G)$ *is connected, then* $G \cong (T(G) \cap K)G)) \times G/(T(G) \cap K(G))$.

Proof. If $K(G) \supset T(G)$, this reduces to the previous theorem, since $K(G) \cap T(G) = T(G)$, and then $T(G)$ is compact.

Hypotheses (1) and (2) are equivalent by theorem 4.18.

REMARK: $T(G) \cap K(G)$ is compact and divisible. Hence $T(G) \cap K(G)$ is an algebraic direct summand of G. If there exists a closed complementary summand of $T(G) \cap K(G)$ in G, $T(G) \cap K(G)$ is a (topological) direct summand.

THEOREM 6.5. *We have* $G \cong (G/(T(G)+K(G))) \times (T(G)+K(G))$, *if any of the following is true:* (1) $T(G)+K(G)$ *is an algebraic direct summand of* G, (2) $T(G)+K(G)$ *is divisible,* (3) $T(G)$ *is divisible, and* (4) G *is divisible.*

Proof. If $T(G)+K(G)$ is an algebraic direct summand, then the conclusion holds by the remark (*).

If $T(G)+K(G)$ is divisible then $T(G)+K(G)$ is an algebraic direct summand; thus (2) implies (1).

If $T(G)$ is divisible, then $K(G)+T(G)$ is divisible since $K(G)$ is divisible. This reduces (3) to the case (2).

Since $T(G)$ is a pure subgroup of G, if G is divisible then, by theorem 4.1, $T(G)$ is divisible. Hence case (4) reduces to case (3).

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