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ON THE EPIMORPHISM THEOREM

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1. Introduction.

F. Trèves introduced the concept of presurjectivity and showed that if u is a continuous linear map on a Fréchet space into a Fréchet space, then u is an epimorphism if and only if u is presurjective. (cf. Trèves [1]) We shall generalize the situation a little further and investigate the relation between presurjectivity and epimorphism of a continuous linear map from $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Omega)$. We shall use the fact that $\mathcal{D}'(\Omega)$ is isomorphic to the complete Hausdorff space of all the regular sections on the set of all the Fréchet spectrums on $\mathcal{D}'(\Omega)$. (cf. Kim [1])

2. Definitions.

We shall denote $\mathcal{D}'(\Omega)$ by E. F-Spec E will denote the set of all Fréchet spectrums on E. When u is a continuous linear map from E into E, we define u_* from F-Spec E into itself such that $(u_*q)_i(v)=q_i(uv)$ for any q in F-Spec E, for any v in E and for any $i=1, 2, 3, \cdots\cdots$.

When p and q are Fréchet spectrums on E such that $q\cdot u\leqq p$, then u regarded as a linear map $E_p\longrightarrow E_q$ is continuous; by extending to the completions, it defines, a continuous linear map

$$u_q^p : \hat{E}_p\longrightarrow\hat{E}_q.$$

When $p=q\cdot u$, we shall write u_q rather than u_q^{u*q}. u_q is an isometry from $\hat{E}_{u*q}$ into $\hat{E}_q$.

DEFINITION. Let $u : E\longrightarrow E$ be a continuous linear map. We conider the commutative diagram:

$$\begin{array}{ccccc} E & \xrightarrow{u} & \operatorname{Im} u & \xrightarrow{j} & E \\ \downarrow{\scriptstyle\phi} & & & \nearrow{\scriptstyle\bar{u}} & \\ E/\ker\bar{u} & & & & \end{array}$$

u is a *homomorphism* if $\bar{u}$ is an open mapping. A homomorphism u is an *epimorphism* if u is surjective.

DEFINITION. A subset A of F-Spec E is *equicontinuons* if there exists p in F-Spec E such that $p\geqq q$ for any q in A.

DEFINITION. We say that a mapping : $E\longrightarrow E$ is *presurjective* if for every equicontinuous subset A of F-Spec E and every y in E there exists a regular section s over A such that

$$us=w_B(y) \qquad B=u_*^{-1}(A)$$

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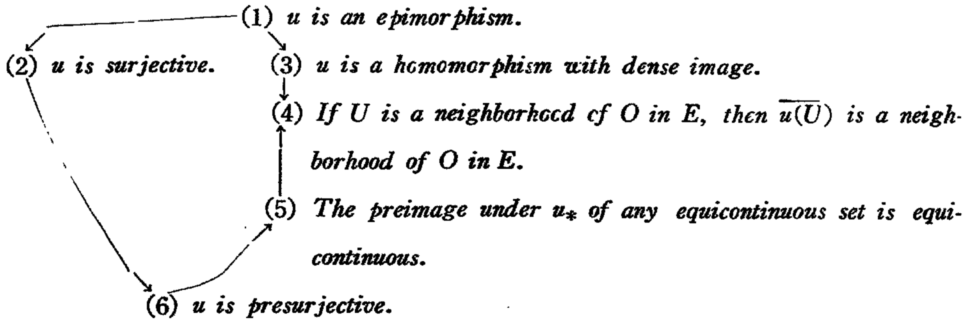
where $us(q)=u_q(s(u_*q))$ for any q in B.

3. Propositions.

PROPOSITION 1. *Let u be a continuous linear map from F into E. Then the following are equivalent: (a) $\operatorname{Im} u$ is dense in E and (b) for any q in F-Spec E $u_q : \hat{E}_{u*q} \to \hat{E}_q$ is surjective.*

Proof. (a) is equivalent with saying that for any q in F-Spec E $\operatorname{Im} u$ is dense in E_q, i.e., $u_q(w_{u*q}(E))$ is dense in $w_q(E)$. As $w_q(E)$ is dense in $\hat{E}_q$, we see that (a) is equivalent to saying that $u_q(w_{u*q}(E))$ is dense in $\hat{E}_q$. As $w_{u*q}(E)$ is dense in E_{u*q}, this is eqivalent to saying that $\operatorname{Im} u_q$ is dense. But u_q is an isometry. Hence $\operatorname{Im} u_q$ is dense if and only if u_q is surjective.

PROPOSITION 2. *Let u be a continuous linear map from E into E. Then the following implications hold.*



Proof. (1) clearly implies (2) and (3).

(2)⟶(6)

If u is surjective, to every y in E we can choose x in E such that $u(x)=y$. Then whatever the subset A of F-Spec E is, we have

$$uw_A(x)=w_B(y) \qquad B=u_*^{-1}(A).$$

(3)⟶(4)

Let U be a neighborhood of O in E. Then $u(U)$ is a neighborhood of O in $\operatorname{Im} u$. Hence there exists V, a neighborhood of O in E such that

$$V \cap \operatorname{Im} u \subseteq u(U).$$

We may take V open. Then if $\operatorname{Im} u$ is dense, $\bar{V}=\overline{V \cap \operatorname{Im} u} \subseteq \overline{V \cap \operatorname{Im} u} \subseteq \overline{u(U)}$. Hence $\overline{u(U)}$ is a neighborhood of O in E.

(6)⟶(5)

Let A be an equicontinuous subset of F-Spec E and B be its preimage under u_*. There exists a Fréchet spectrum p_0 such that $q \leqq p_0$ for any q in A. Let y in B be arbitrary and s be a regular section over A such that $us=w_B(y)$. For any q in B we have

$$(u_*q)_i(s(u_*q)) \leqq p_{0i}(s(p_0)):$$

But $us(q)=u_q(s(u_{*q}))=w_g(y)$.
Also $q_i(us(q))=(u_*q)_i(s(u_*q))=q_i(w_q(y))$ since u_q is an isometry:
Also $q_i(w_q(y))=q_i(y)$.
Finally, for every y in E $q_i(y)\leqq p_{0i}(s(p_0))$.
Let $q_{0i}=\sup_{q\in B} q_i$. Then since E is barrelled, q_{0i} is a continuous seminorm. Let $q_0=(q_{01}, q_{02}, \cdots\cdots)$. Then for any q in B, $q\leqq q_0$, which says that B is equicontinuous.
(5)$\longrightarrow$(4)
It suffices to prove that $\overline{u(U)}$ is a barrel whenever U is the closed unit ball of a continuous seminorm p_0 on E. Let us identify p_0 on E. Let us identify p_0 with a Fréchet spectrum $(p_0, p_0, \cdots\cdots)$. Let B be the subset of F–Spec E consisting of the Fréchet spectrums q such that $u_*q\leqq p_0$. We have $q_{0i}(y)=\sup_{q\in B} q_i(y)<+\infty$ for all y in E. Note that $q_{0i}=q_{0j}$ for any i and j. Let $q_0=q_{0i}$. We claim that $u(U)$ is equal to the closed unit ball B_{q_0} of the seminorm q_0. This will complete the proof.
Let x in E be such that $p_0(x)\leqq 1$; then $q(u(x))\leqq 1$ for all q in B. Hence $u(U)$, and therefore $\overline{u(U)}$, is contained in B_{q0}. Let y in E belong to the complement of $\overline{u(U)}$. There is a continuous seminorm q such that $q\leqq 1$ on $\overline{u(U)}$ and $q(y)>1$; this shows that q belongs to B and y is not in B_{q_0}. Thus $u(U)=B_{q_0}$.

References

F. Trèves [1]: *Locally Convex Spaces and Linear Partial Differential Equations.* Springer–Verlag 1967.

J. Kim [1]: *Distributions and Partial Differential Equations.* Jour. of Korean Math. Soc., **11** (1974).

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