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On Matroids

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1. Introduction

Until the beginning of this century matroid theory was little more than a patch work of isolated results. The discipline is still very young if it has now attained the status of a distinctive discipline. A few results can be traced back half a century, but a majority of theorems were discovered during the last ten or fifteen years, and many during the last five. The current rate of progress is exceptionally rapid and the subject is still in a state of flux.

It has also been proved recently that a transversal matroid is binary if and only if it is graphic. Related to this is the main theorem treated here that the circuit matroid $M(G)$ of a graph G is transversal if and only if G contains no subgraph homeomorphic to K_4 or C_2^n for some n.

Our graph-theoretic terminology is standard and is based essentially on the book of Harary [1]. For the matroid and transversal theory we refer to the book of Wilson [2].

Throughout this note E will denote a finite set, and $|X|$ denotes the cardinality of the set X. I will denote a finite index set, $(A_i : i \in I)$ denotes a family of sets indexed by I, and $\{A_i : i \in I\}$ is the set consisting of the distinct members of the family $(A_i : i \in I)$. If A is a set and x an element we often write $A+x$ to denote $A \cup \{x\}$.

If A is a subsets of E we write $\bar{A}$ for $E-A$.

2. Basic Concepts

DEFINITION 2.1 *A matroid* $M=(E, B)$ consists of a nonempty finite set E, together with a non-empty family B of subsets of E (called *bases*) satisfyting the following properties:

(i) No base properly contains another base.

(ii) If B_1 and B_2 are bases, and x is an element of B_1, then there exists an element y of B_2 such that B_1-x+y is also a base.

We can easily deduce from these two properties that any two bases of a matroid contain the same number of elements. The common cardinality of the bases is the ***rank*** $r(M)$ of the matroid M. Let $\mathfrak{B}(M)$ denote the collection of bases of matroid M.

If M is matroid on a set E, we say that a subset A of E is an ***independent set*** of M if A is contained in some base of M. It follows that the bases of M are precisely the maximal independent sets. If X is any subset of E, all maximal independent subsets of X have the unique cardinality, called the ***rank*** of X, and denoted by $r(X)$.

Any subset which is not independent is called ***dependent***, and the minimal dependent sets are called the *circuits* of M: we denote the collection of circuits of M by $\mathfrak{C}(M)$. Two matroids M_1 on E_1 and M_2 on E_2 are ***isomorphic*** if there is a bijection $f: E_1 \to E_2$ which preserves independence.

DEFINITION 2.2 If M is a matroid on E, we define the *dual matroid* M^* to be the matroid on E whose bases are precisely the complements of the bases of M; in other words B^* is a base of M^* if and only if $E-B^*$ is a base of M.

It is not difficult to check that this does in fact define a matroid. It follows immediately from this definition that every matroid has a dual, and that this dual is unique. Moreover, it is clear that the double-dual M^{**} is equal to M. Let us define some 'co-notation'. If M^* is the dual of M, we define some 'co-notation'. If M^* is the dual of M, we define a *cocircuit* of M to be a circuit of M^*. Similarly we define a *cobase* of M to be a base of M^* and so on. The reason for introducing these extra definitions is that we need now deal only with the matroid M, instead of dealing with both M and M^*.

We frequently make use of the following matroid theorems. [7] [9]

LEMMA 2.1 *A cocircuit C^* of a matroid M on a set E is precisely a minimal non-empty subset of E having a non-empty intersection with every base of M.*

LEMMA 2.2 *Let M be a matroid on a set E. If C_1, C_2 are two circuits of M with $x \in C_1 \cap C_2$ $y \in C_2 - C_1$, there exists a circuit C_3 such that $y \in C_3 \subseteq (C_1 \cup C_2) - x$.*

LEMMA 2.3 *If B is any base of M and $\bar{B} = \{x_1, \cdots x_m\}$, there is a unique circuit C_i containing only x_i and some elements of B.*

We may interprete Lemma 2.3 as follows: let B be a base and $x \in \bar{B}$, then $B+x$ contains a unique circuit of M (which we denote by C_x). Hence we have the following definition. [6]

DEFINITION 2.3 Let B be a base, and let $x_1, \cdots, x_m$ be the elements of $\bar{B}$, then $B+x_1, \cdots B+x_m$ contain unique circuits $C_{x_1}, \cdots, C_{x_m}$, respectively. These are called the *fundamental set of circuits* associated with B, and shall sometimes be called simply $C_1, \cdots, C_m$. Each circuit C_i is called the *fundamental circuit of x_i in B.*

By duality any base B of M also determines a fundamental set of cocircuits, namely, the fundamental circuits of M^* determined by the base $E-B$ of M^*.

DEFINITION 2.4 The *reduction matroid* $M \times A$ is the matroid on A whose circuits are precisely those circuits of M which are contained in A; similarly, the *contraction matroid* $M \cdot A$ is the matroid on A whose cocircuits are precisely those cocircuits of M which are contained in A.

Reduction and contractions are related by the identity $(M \times A)^* = M^* \cdot A$ We have a special case of contraction, when $A=E-y$. [8]

DEFINITION 2.5 For some element x of M, let $\{x, y\}$ be a cocircuit of a matroid M on a set E, then $M\cdot(E-y)$ is called the *series contraction* of M at y
wherever $\mathfrak{B}(M\cdot E-y)=\{B-y \mid y\in B\in \mathfrak{B}(M)\}$.

DEFINITION 2.6 We can define a matroid on the set of edges of a graph G by taking as bases of the matroid the edges of the various spanning forests of G. This matroid is called the *circuit matroid* of G, and denoted by $M(G)$.

It follows that a set of edges of G is independent if and only if it contains no circuit of G, and that the circuits of the matroid $M(G)$ are precisely the circuits of G.

The graphical significance of the matroid operations of reduction and contraction can be explained as follows. If G is a graph, and A is a subset of $E(G)$, define $G\times A$ to be the graph obtained from G by deleting all edges not in A, and $G\cdot A$ to be the graph obtained by contracting all edges not in A. [2] Two graphs are *homeomorhpic* if each can be obtained from the same graph by a sequence of edge subdivisions.

DEFINITION 2.7 Let B_1 and B_2 be two bases of a matroid M. An *exchange ordering* of B_1 and B_2 is a bijective mapping $\sigma: B_1\rightarrow B_2$ such that, for all $x\in B_1$, both $B_1-x+\sigma(x)$ and $B_2-\sigma(x)+x$ are bases of M. A matroid M is *base-orderable* if there is an exchange ordering for every pair of bases of M. [5]

DEFINITION 2.8 Let M_1, M_2 be matroids on disjoint sets E_1, E_2 respectively, the *sum* M_1+M_2 of M_1 and M_2 is the matroid on $E_1\cup E_2$ with bases

$$\mathfrak{B}(M_1+M_2)=\{B_1\cup B_2 \mid B_1\in \mathfrak{B}(M_1),\ B_2\in \mathfrak{B}(M_2)\}.$$

3. Transversal Matroids

DEFINITION 3.1 Let $\mathfrak{A}=(A_1, \cdots, A_n)$ be a family of subsets of a finite set E. A subset $X=\{x_1, \cdots, x_k\}$ of distinct elements of E is a *partial transversal* of A is there exists a subfamily $(A_{i_1}, \cdots, A_{i_k})$ of such that $x_j\in A_{ij}$ $(1\leqq j\leqq k)$. (x_j is said to represent A_{ij} in X)

The maximal partial transversals are then the bases of a matroid on E. When these maximal partial transversals have cardinality n we call them *transversals* of A. The basis result linking transversal theory and matroids is the following:

If $\mathfrak{A}=(A_1, \cdots, A_n)$ is a family of subsets of a finite set E, the set of partial transversals of $\mathfrak{A}$ is the collection of independent sets of a matroid on E.

DEFINITION 3.2 A matroid M on E is called a *transversal matroid* if there exists some family $\mathfrak{A}$ of subsets of E such that $\mathfrak{B}(M)$ is the family of maximal partial transversals of $\mathfrak{A}$.

When M is a transversal matroid such that the independent sets of M are the partial transversals of a family of $\mathfrak{A}=(A_1, \cdots, A_n)$, we call $(A_1, \cdots, A_n)$ a *transversal presentation* or *presentation* of M, and write $M=M(A_1, \cdots, A_n)$.

For the proof of the main theorem we require several lemmas. [3] [4]

LEMMA 3.1 *Let M be the transversal matroid $M(A_1, \cdots, A_r)$ of rank r and let D be a transversal of $(A_2, \cdots, A_r)$ such that $X=D\cap A_1$ has minimum cardinality.*
Then $M=M(A_1-X, A_2, \cdots, A_r)$.

Proof. Let $B=\{b_1, b_2, \cdots, b_r\}$ be a transversal of $(A_1, \cdots, A_r)$. Lemma 3.1 is proved if we show that B is a transversal of $(A_1-X, A_2, \cdots, A_r)$. Let $D=\{d_2, \cdots, d_r\}$ where $d_i\in A_i$ $(2\leqq i\leqq r)$.

$(1)_1$ Suppose $b_1\in A_1-X$, it is obviously true.

$(2)_1$ Suppose $b_1\notin A_1-X$, then $b_1\in X=D\cap A_1$.

Hence $b_1\in D$. We may without loss of generality put $b_1=d_2$.

$(1)_2$ If $b_2\in A_1-X$ then $b_1=d_2\in A_2$ and $B=\{b_2, b_1, b_3, \cdots, b_r\}$ is a transversal of $(A_1-X, A_2, \cdots, A_r)$.

$(2)_2$ If $b_2\notin A_1-X$, then $b_2\in D$ or $b_2\notin A_1\cup D$

Suppose $b_2\notin A_1\cap D$, then $D'=\{b_2, d_3, \cdots, d_r\}$ is a transversal of $(A_2, \cdots, A_r)$. Since $b_2\notin A_1$, $d_2=b_1\in A_1$,

$$|D'\cap A_1|<|D\cap A_1|$$

which is a contradiction. Therefore $b_2\in D$. Also $b_2\neq d_2$ since $d_2=b_1\neq b_2$.

Hence let $b_2=d_3$, say.

Now consider b_3.

$(1)_3$ If $b_3\in A_1-X$, then $B=\{b_3, b_1, b_2, b_4, \cdots, b_r\}$ is a transversal of $(A_1-X, A_2, \cdots, A_r)$.

$(2)_3$ If $b_3\notin A_1-X$, then $b_3\in D$ or $b_3\notin A_1\cup D$.

Suppose $b_3\notin A_1\cup D$, then $D''=\{b_2, b_3, d_4, \cdots, d_r\}$ is a transversal of $(A_2, \cdots, A_r)$ if $d_3\in A_1$ then $|D''\cap A_1|<|D\cap A_1|$. This is a contradiction. If $d_3\notin A_1$, then $b_2=d_3\notin A_1$ and $b_2=b_1\in A_1$ and so $|D''\cap A_1|<|D\cap A_1|$.

This is a contradiction. Hence $b_3=d_4\in D$.

Carrying on this way see that either

$$B=\{b_i, b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_r\}$$

is a transversal of $(A_1-X, A_2, \cdots, A_r)$ for some i or we get the contradiction that $b_i=d_{i+1}$ for $1\leqq i\leqq r$. This completes the proof.

LEMMA 3.2 *If M is a transversal matroid of rank r, with presentation $M=M(A_1, \cdots, A_r)$, then there exist distinct cocircuits C_i^* $(1 \leqq i \leqq r)$ such that for some distinct $i_1, i_2, \cdots, i_r$*

$$C_j^* \subseteq A_{i_j} \ (1 \leqq j \leqq r), \text{ and } M=M(C_1^*, \cdots, C_r^*).$$

Proof. Let D be a transversal of $(A_2, \cdots, A_r)$ such that $X=D \cap A_1$ has minimum cardinality. Then $\quad (A_1-X) \cap D=\phi$

and hence for any $y \in A_1-X$, $D+y$ is a base of M. Let $B=\{b_1, \cdots, b_r\}$ be a base of M. Put $D'=\{b_2, \cdots, b_r\}$. Then D' is a transversal of $(A_2, \cdots, A_r)$. Hence $|D' \cap A_1| \geqq |D \cap A_1|$. Therefore $B \cap (A_1-X) \neq \phi$.

This means that A_1-X is minimal set intersecting every base of M, and so A_1-X is a cocircuit of M. Applying this procedure to A_i $(1 \leqq i \leqq r)$ and noticing that for any matroid $M=M(A_1, \cdots, A_r)$, if $A_i=A_j$, then there exists $A_i' \subseteq A_i$, $A_i' \neq A_i$ such that $M=M(A_1, \cdots, A_i', \cdots A_j, \cdots, A_r)$, shows that the cocircuit presentation must be distinct.

LEMMA 3.3 *The transversal matroids $M(A_1, \cdots, A_r)=M(A_1+x, \cdots, A_r)$ if and only if X is contained in every transversal of $(A_2-A_1, A_3-A_1, \cdots, A_r-A_1)$.*

Proof. We use the proof by induction on $|X|$.

Suppose $X=\{x\}$. Let x be ccontained in every transversal of $(A_2-A_1, \cdots, A_r-A_1)$. Then x is a cocircuit of $M(A_2-A_1, \cdots, A_r-A_1)$. Hence every transversal of $(A_2, \cdots, A_r)$ intersects A_1+x. Choose a transversal, B_1 say, of $(A_2, \cdots, A_r)$ such that $|B_1 \cap (A_1+x)|$ is a minimum.

(1) If $x \in B_1$, then by Lemma 3.1

$$M(A_1+x, \ A_2, \cdots, A_r)=M((A_1+x)-(B_1 \cap A_1+x), \ A_2, \cdots, A_r)$$
$$=M(A_1-B_1, \ A_2, \cdots, A_r)$$

Thus $\quad M(A_1+x, \ A_2, \cdots, A_r)=M(A_1, \cdots, A_r)$.

(2) Suppose $x \notin B_1$. Since x belongs to every transversal of $(A_2-A_1, \cdots, A_r-A_1)$, it is clear that $B'=B_1-(A_1+x)$ is a partial transversal of $(A_2-A_1, \cdots, A_r-A_1)$.

By the choice of B_1, it is easy to see that B' is an miximal partial transversal of $(A_2-A_1, \cdots, A_r-A_1)$. Hence B' is independent in $M'=M(A_2, \cdots, A_r)$ and we can augment B' to a base B_2 of M'. Clearly

$$|B_2 \cap (A_1+x)|=|B_1 \cap (A_1+x)|$$

Now apply (1) to B_2.

Conversely suppose $M(A_1+x, \cdots, A_r)=M(A_1, \cdots, A_r)$. Consider transversal D of $(A_2-A_1, \cdots A_r-A_1)$. D is a partial transversal of $(A_2, \cdots, A_r)$ and hence $D+x$ is a partial transversal of $(A_1+x, \cdots A_2, \cdots, A_r)$. Since $M(A_1+x, \cdots, A_r)=M(A_1, \cdots, A_r)$, $D+x$ is a partial transversal of $(A_1, \cdots, A_r)$. But $(D+x) \cap A_1=\phi$ and so $D+x$ is a partial transversal of $(A_2-A_1, \cdots, A_r-A_1)$.

Since D is a transversal, $D=D+x$, that is $x \in D$. Hence x is contained in every transversal of $(A_2-A_1, \cdots, A_r-A_1)$.

Next, when $X=\{x, y\}$ we shall complete Lemma 3.3.

If $M=M(A_1, \cdots, A_r)=M(A_1+x,\ A_2, \cdots, A_r)=M(A_1+y,\ A_2, \cdots, A_r)$,

then $M=M(A_1+x+y,\ A_2, \cdots, A_r)$. Hence it is obvious.

LEMMA 3.4 *Let M be a transversal matroid.*

Then any reduction $M \times A$ of M is also transversal.

Proof. Let $\mathfrak{A}=(A_1, \cdots, A_r)$ be a presentation of M. Put $\mathfrak{A}_A=(A_1 \cap A, \cdots, A_r \cap A)$. Then $\mathfrak{A}_A$ is a presentation of $M \times A$.

THEOREM 3.1 *Let M be a transversal matroid on E, and let x and y be a cocircuit of M. Then $M \cdot (E-y)$, the series contraction of M at y, is also transversal.*

Proof. Let $r(M)=r$. By Lemma 3.2, M has as a presentation some family $\mathfrak{C}^*$ of its cocircuits; There are r cocircuits in $\mathfrak{C}^*$ and all are distinct. Such a presentation, although not necessarily unique, is minimal.

Let $\mathfrak{C}^*=(C_1^*, \cdots, C_r^*)$, and put $A=E-\{x, y\}$.

Then, by Lemma 3.4, $\mathfrak{C}_A^*=(C_1^* \cap A, \cdots, C_r^* \cap A)$ is a presentation of $M \times A$. Now, since $\{x, y\}$ is a cocircuit of M, $M \times A$ has rank $r-1$, and so some $r-1$ of the sets in $\mathfrak{C}_A^*$ give a presentation of $M \times A$. So we may assume that $\mathfrak{C}_A^{*\prime}=(C_2^* \cap A, \cdots, C_r^* \cap A)$ is a presentation of $M \times A$. Since every transversal of $\mathfrak{C}_A^{*\prime}$ contains all of $C_1^* \cap A$, and since some transversal of $(C_2^*, \cdots, C_r^*)$ does not interesect C_1^* at all, it follows that C_1^* can contain at most two elements.

Suppose C_1^* contains just one element z. then, since $\{z\}$ is a cocircuit of M, z is neither x nor y. Now every transversal of $\mathfrak{C}_A^{*\prime}$ contains $C_1^* \cap A=\{z\}$, and so z is contained in some cocircuit C_j^* other than C_1^*. But this is impossible since the members of $\mathfrak{C}^*$ are distinct cocircuits. Hence C_1^* has exectly two elements.

Let $C_1^*=\{\omega, z\}$. We show that $C_1^*=\{\omega, z\}=\{x, y\}$.

Suppose first that $\{\omega, z\} \cap \{x, y\}=\phi$. Let

$$U=E-\{\omega, z\}, \quad \bar{V}=E-\{x, y, \omega, z\}.$$

Then since $r(M \times A)=r-1$, and since both $\{\omega\}$ and $\{z\}$ are cocurcuits of $M \times A$, $r(M \times \bar{V})=r-3$. But, since $C_1^*=\{\omega, z\}$ is a cocircuit of M, $r(M \times U)=r-1$. Hence both $\{x\}$ and $\{y\}$ are cocircuits of $M \times U$. In other words x and y are in every transversal of $(C_2^*-C_1^*, \cdots, C_r^*-C_1^*)$ and so, by Lemma 3.3 $(C_1^* \cup \{x, y\}, C_2^*, \cdots, C_r^*)$ is a presentation of M. Now since $r(M \times \bar{V})=r-3$ every transversal of $(C_2^*, \cdots, C_r^*)$ intersects $C_1^* \cup \{x, y\}$ in at least two elements. But there is some transversal of $\mathfrak{C}_A^{*\prime}$ which intersects this set in $\{\omega, z\}$. Therefore, by Lemma 3.1, $(\{x, y\}, \cdots$

$C_2^*, \cdots, C_r^*)$ is a presentation of M.

If $\{\omega, z\} \cap \{x, y\} = \{x\}$, say,

then y is in every transversal of $(C_2^*-C_1^*, \cdots, C_r^*-C_1^*)$. Hence $(C_1^*+y, C_2^*, \cdots, C_r^*)$ is a presentation of M. Now every transversal of $(C_2^*, \cdots, C_r^*)$ intersects C_1^*+y. But, as above, there is a transversal which does not contain x or y. Therefore $(\{x, y\}, C_2^*, \cdots, C_r^*)$ is a presentation of M.

We may therefore now assurme that $C_1^*=\{x, y\}$.

For $2 \leqq i \leqq r$, write

$$D_i^* = \begin{cases} C_i^* & \text{If } C_i^* \cap \{x, y\} = \phi \\ (C_i^* \cap A) + x & \text{otherwise} \end{cases}$$

We shall show that $\mathfrak{D}^* = (D_2^*, \cdots, D_r^*)$ is a presentation of $M \cdot (E-y)$

Let B be a base of M, with y in B. Then B is a transversal of $\mathfrak{C}^*$. If y represents C_1^*, clearly. $B-y$ is a transversal of $(C_2^*, \cdots, C^*_r)$ and hence of $(D_2^*, \cdots, D_r^*)$. If x represents C_1^* in B, y must represent some other set C_j^* in B. Then $x \in D_j^*$, and so $B-y$ is a transversal of $(D_2^*, \cdots, D_r^*)$. Hence in either case, $B-y$ is a base of a matroid $M(D_2^*, \cdots, D_r^*)$ on $E-y$.

Conversely, suppose B^* is a base of a matroid $M(D_2^*, \cdots, D_r^*)$ on $E-y$. Then B^* is a transversal of $(D_2^*, \cdots, D_r^*)$. If $x \notin B^*$, B^* is a transversal of $(C_2^*, \cdots, C^*_r)$ and so B^*+y is a transversal of $(C_1^*, \cdots, C_r^*)$. If $x \in B^*$, suppose x represents D_j^* in B^*. Then either $x \in C_j^*$ or $y \in C_j^*$. If $x \in C_j^*$, B^* is a transversal of $(C_2^*, \cdots, C_r^*)$ and so B^*+y is a transversal of $(C_1^*, \cdots$ $C_r^*)$ If $y \in C_j^*$, B^*-x+y is a transversal and again B^*+y is a transversal of $(C_1^*, \cdots, C_r^*)$. Hence it is always the case B^*+y is a base of M. It follows that $M \cdot (E-y)$ is a matroid on $E-y$.

4. Main theorem

The main theorem characterizes those graphs whose circuit matroids are transversal. We begain with several lemmas.

LEMMA 4.1 *Let $M(K_4)$ be a circuit matroid of a complete graph K_4. Then $M(K_4)$ is not transversal.*

Proof. It is easy to show that $M(K_4)$ is not base-orderable. Since every transversal matrodi is base-orderable, $M(K_4)$ is not transversal. [5]

LEMMA 4.2 *Let C_k^2 is the graph obtained from the circuit of length k on replacement of each edge by a pair of parallel edges. Let $M(C_k^2)$ be a circuit matroid of C_k^2. Then $M(C_k^2)$ is not transversal for $k>2$.*

Proof. Suppose the lemma is false, and let $(C_1^*, \cdots, C^*_{k-1})$ be a cocircuit presentation of

$M(C_k^2)$. Label the edges of C_k^2 with the set $\{1, 1', 2, 2', \cdots, k, k'\}$ so that parallel edges are assigned labels with the same number. Then each cocircuit of $M(C_k^2)$, and hence each C_i^*, is of the form $\{m, m', n, n'\}$. Since $k>2$, there is some j, $1 \leqq j \leqq k$, such that $\{j, j'\}$ is contained in at least two of the cocircuits C_i^*. But then $\{j, j'\}$ a circuit in C_k^2, is a partial transversal of $(C_1^*, \cdots C^*_{k-1})$, and we have a contradiction.

LEMMA 4.3 *Let M_1, M_2 be transversal matroids on disjoint sets E_1, E_2 respectively. Then the sum M_1+M_2 of M_1 and M_2 is transversal.*

Proof. Let $(A_1, \cdots, A_m)$ be a presentation of M_1, and let $(B_1, \cdots, B_n)$ be a presentation of M_2. Then $(A_1, \cdots, A_m, B_1, \cdots, B_n)$ is a presentation of M_1+M_2. [4]

LEMMA 4.4 *Let G be a black of order greater than two, with no subgraph homeomorphic to K_4 or C_k^2 $(k>2)$. Then G contains a vertex adjacent to exactly two vertices, and joined by just one edge to one of these vertices.*

Proof. The proof is by induction on the order of G. The lemma clearly holds if G has order three. Suppose now that the order of G is $N>3$. There is some vertex u in G adjacent to exactly two vertices, v and w. Let us assume that u is joined to v by edges $e_1, \cdots, e_m$, and to w by edges $f_1, \cdots, f_n$. If either m or n is one, u is the required vertex. Otherwise both m and n are at least two.

(1) $(v, w) \notin E(G)$. Let $G' = G \cdot (E(G) - \{f_1, \cdots, f_n\})$. Then G' is a block of order $N-1$ and contains no subgraph homeomorphic to K_4 or C_k^2 $(k>2)$ since G does not. Hence, by the induction hypothesis, there is a vertex $x(\neq u)$ in G' adjacent to exactly two vertices of G', and joined by just one edge to one of these vertices. The vertex x has same property in G.

(2) $(v, w) \in E(G)$. Since G contains no homeomorph of C_3^2 there is only one edge g joining v and w, and moerover, if H denotes the graph obtained by deleting the vertex u its incident edges $e_1, \cdots, e_m$, $f_1, \cdots, f_n$, and the edge g. If H is connected, then G contains a subgraph homeomorphic to C_3^2. Hence H must be disconnected, with v and w in different components. Therefore v and w are cut-vertices of G. This is a contradiction since G is a block.

THEOREM 4.1 *Let G be a finite graph. Then $M(G)$, the circuit matroid of G, is transversal if and only if G contains no subgraph homeomorphic to K_4 or C_k^2 $(k>2)$.*

Proof. Suppose there exists a subgraph H of G which is homeomorphic to K_4. Then H can be obtained from G by deleting all the edges not in $E(H)$. Hence, by Lemma 3.4, $M(H)$ is transversal. If H is a complete graph on four vertices, then $M(H)=M(K_4)$ is transversal. This is impossible by Lemma 4.1. If H is the graph obtained from K_4 by inserting a new vertex into an edge of K_4, then K_4 is obtained from H by contracting one of new edges. Hence, by theorem 3.1 $M(K_4)$ is transversal. It is impossible by Lemma 4.1. Therefore G contains no subgraph homeomorphic to K_4. It can be shown by the same method that G contains no subgraph homeomorphic to C_k^2.

Conversely, suppose G contains no subgraph homeomorphic to K_4 or C_k^2 ($k>2$). Since the circuit matroid of a graph is the sum of the circuit matroids of its blocks, it follows from Lemma 4.3 that we can assume that G is a block. We shall prove, by induction on the order of G, that $M(G)$ is transversal. This is trivially graphs of orders one and two. Suppose it is true for graphs of order less than N, and let G have order $N>2$. By Lemma 4.4, there is a vertex u in G such that u is adjacent to exactly two vertices v and w, and is joined to v by one edge e and to w by edges $f_1, \cdots, f_n$.

Let $G' + e = G \times (E(G) - \{f_1, \cdots, f_n\})$.

Then G' is a block of order $N-1$ and G' contains no subgraph homeomorphic to K_4 or C_k^2 ($k>2$) since G does not. Hence $M(G')$ is transversal. Let $M(G')$ have the presentation $(A_1, \cdots, A_r)$ Put $A_{r+1} = \{e, f_1, \cdots, f_n\}$. Then, since A_{r+1} is a cocircuit of $M(G)$,

$$M(G) = M(A_1, A_2, \cdots, A_r, A_{r+1})$$

Hence $M(G)$ is a transversal matroid.

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논 문 개 요

Matroid 와 transversal theory 에 관한 연구가 활발하게 진행되고 있다. 이 논문은 circuit matroid 가 transversal 이 되기 위한 필요 충분 조건을 지적하고 이를 증명한 것이다. 곧 어떤 graph 의 circuit matroid 가 transversal 이 되기 위한 필요 충분 조건은 그 graph 가 graph K_4 거나 C_k^2 에 homeomorphic 인 subgraph 를 포함하지 않은 것이다. 증명의 방법으로는 주로 matroid 의 cocircuit 의 성질을 사용하여 transversal Matroid 의 presentation 을 적절하게 적용하고 graph 의 reduction 과 contraction 을 이용한다.