韓國數學敎育學會誌〈數學敎育〉
1975. 6 Vol. XIII No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1975. Vol. XIII No. 3

Rational Extentions of Modules and D Rings

by

Jung Wan Nam and Kang-Joo Min

Gyeong Sang National University, JinJu, Korea.

§1. Introduction.

This paper is concerned with the study of modules whose lattice of submodules is distributive and, in particular, the study of rings R such that R^R is a D module. In §2 we consider the basic properties of D modules and D rings. In §3 we consider the rational extensions of modules over a left D ring and we obtain the following result. If R is a left D ring, then every left R-module is rationally complete.

§2. D rings and D modules.

Throughout this paper, R will denote an associative ring with identity 1 and each module M will be a unitary left R-module. L(M) will denote the lattice of submodules of M and E(M) denote the injective hull of M.

DEFINTION 2.1. (a) M is said to be a D module if L(M) is a distributive lattice. That is,

1) for all $A, B, C \in L(M)$, $A\cap(B+C)=(A\cap B)+(A\cap C)$ or equivalently

1)′ for all $A, B, C \in L(M)$. $A+(B\cap C)=(A+B)\cap(A+C)$.

b) R is said to be a left (right) D ring if $R^R({}^RR)$ is a D module.

PROPOSTITION 2.2. Suppose that M is a D module if and only if $\mathrm{Hom}_R(A/A\cap B, B/A\cap B)=0$ for all $A, B \in L(M)$.

Proof. A lattice is distributive if and only if relative complements are unique [3]. If $A\cap B=0$, then there is a bijection between $\mathrm{Hom}_R(A, B)$ and the set of complements of B in A+B. Thus working modulo $A\cap B$, the complements of B in A+B relative to $A\cap B$ are in one to one correspondence with $\mathrm{Hom}(A/A\cap B, B/A\cap B)$. The result follows.

PROPOSTION 2.3. Every idempotent of a left D ring is central.

Proof. If R is a left D ring and $e\in R$ is an idempotent, then $R=Re\oplus(1-e)$ and $eR(1-e)=\mathrm{Hom}((Re, R(1-e))=0$. Similarly $(1-e)Re=0$ and so e is central.

PROPOSTITION 2.4. M is a D module if and only if for every module P and $f\in \mathrm{Hom}(P, M)$, $f^{-1}(A+B)=f^{-1}(A)+f^{-1}(B)$ for all $A, B\in L(M)$.

Proof. Let C be any submodule of M and $f: C\to M$ be the inclusion map. Then $(A+B)\cap C= f^{-1}(A+B)=f^{-1}(A)+f^{-1}(B)=A\cap C+B\cap C$. $(\Rightarrow) f^{-1}(A)+f^{-1}(B)=f^{-1}ff^{-1}(A)+f^{-1}ff^{-1}(B)= f^{-1}(ff^{-1}(A)+ff^{-1}(B))=f^{-1}[A\cap f(P)+B\cap f(P)]=f^{-1}(A+B)$.

DEFINITION 2.5. A ring R is said to be left subcommutative if every left ideal of R is a two-ideal.

PROPOSITION 2.6. If R is a left artinian D ring, then R is left subcommutative.

Proof. Let M be a left ideal of R. Suppose that $Mr\not\subseteq M$ for some $r\in R$. Put $A=\sum\{B\in L(M)\mid Br\subseteq B\}$. Clearly A is the largest left ideal of R contained in M and $Ar\subseteq A$. Since $A\neq M$, there is a left ideal X of R such that $A\subseteq X\subseteq M$, $A\neq X$ and X/A is a simlpe left R-module.

(Notation. Let $r \in R$. $Ar^{-1} = \{x \mid xr \in A\}$)

Then $A \subseteq Ar^{-1} \subseteq Xr^{-1}$, so that $A \subseteq X \cap Xr^{-1} \subseteq X$. Since $Xr \subsetneq X$ and X/A is simple, $A = X \cap Xr^{-1}$. By an easy calculation using proposition 2.4, $X = (X \cap Xr^{-1}) + (X \cap Xr)$. $X = (X \cap Xr^{-1}) + (X \cap Xr) = A + (X \cap Xr) = A + (X \cap Xr^{-1})r = A + Ar = A$. It is a contradiction.

THEOREM 2.7. R is a semi-perfect left D ring if and only if R is the finite direct product of left valuation rings.

Proof. (⇒) Let R be a semi-perfect ring. Then R has a complete of orthogonal idempotents e_1, e_2, e_3, $\cdots\cdots\cdots e_n$.

Then $R = e_1R \oplus e_2R \oplus \cdots\cdots\cdots \oplus e_nR$. Since idempotents in a left D ring are central, $R = e_1Re_1 \oplus e_2Re_2 \oplus \cdots\cdots\cdots \oplus e_nRe_n$.

Since e_iRe_i is a local ring [2], e_iRe_i is a left valuation ring [4]. Therefore R is the finite direct product of left valuation rings.

(⇐) Since a left valuation ring is a left D ring [4] and any direct product of left D rings is again a left D ring, the theorem is clear.

THEOREM 2.8. Let R be a left D ring. The following assertions are equivalent:

1) R is left perfect,
2) R is right perfect,
3) R is left artinian.

Proof. 3)⇒1) and 3)⇒2) are obvious[1].

1)⇒3) Since a left or right perfect ring is certainly semi-perfect, we can assume, with out loss of generality, that R is a left valuation ring. If R is left perfect, then R has the ascending chain condition on principal left ideals.

But any finitely generated left ideal of a left valuation ring is principal, and so R has the ascending chain condition on finitely generated left ideals. Therefore it follows that R is left noetherian. Hence R is left artinian[1].

2)⇒3) If R is right perfect, then R has the descending chain condition on principal left ideals. Suppose that $A_1 \supset A_2 \supset A_3 \supset \cdots\cdots\cdots$ is a strictly descending chain of left ideals of R.

Choose $a_i \in A_i$ but $a_i \notin A_{i+1}$. Since R is a left valuation ring,. $A_i \supseteq Ra_i \supset A_{i+1}$. Hence we obtain a strictly descending chain of principal left ideals, a contradiction.

§3. Rational extentions of modules over a D ring.

DEFINITION 3.1. A submodule N of M is called large in M (written $N \subseteq M$) and M is called an essential extention of N provided that $N \cap K \neq 0$ for every nonzero submodule K of M.

DEFINITION 3.2. Let N be a submodule of M. M is called a rational extention of N if for each submodule B such that $N \subseteq B \subseteq M$, $f \in \mathrm{Hom}_R(B, M)$ satisfies $f(N) = 0$ if and only if $f = 0$ [2].

DEFINITION 3.3. A module M is rationally complete provided that M has no proper rational extention[2].

NOTATIONS 3.4. Let A be an R-module. If $a \in A$, $(a)^R = \{r \in R \mid ra = 0\}$. C(A) denote the rational completion of A.

PROPOSITION 3.5. A module M is rationally complete if and only if $M = C(M)$.

PROPOSITION 3.6. Let A be any simple left R-module.

Let $S(A) = \{(x)^R \mid 0 \neq x \in E(A)\}$. Then $x \in C(A)$ if and only if $(x)^R$ is maximal in $S(A)$.

Proof. If $0 \neq x \in C(A)$ and $(x)^R$ is not maximal in $S(A)$, then there is a $0 \neq y \in E(A)$ such that

$(x)^R \subseteq (y)^R$ and there is an $r' \in R$ such that $r'x \neq 0$ but $r'y = 0$. Define $\phi : Rx \longrightarrow Ry$ by $\phi(rx) = ry$ for all $r \in R$. Then ϕ can be extended to $\bar{\phi} \in \mathrm{Hom}(E(A), E(A))$ and $\bar{\phi}(r'x) = \phi(r'x) = r'y = 0$. Thus $0 \neq r'x \in \mathrm{Ker}\bar{\phi}$ and therefore $\bar{\phi} \neq 0$.

Then since A is simple and $A \cap \mathrm{Ker}\bar{\phi} \neq 0$, $\bar{\phi}(A) = 0$. Thus $C(A) \subseteq \mathrm{Ker}\bar{\phi}$ and in particular $\bar{\phi}(x) = 0$. Hence $Ry = \phi(Rx) = R\phi(x) = 0$.

Thus it follows that $y = 0$ which contradicts the original assumption.

Thus $(x)^R$ is maximal in $S(A)$.

($\Rightarrow$) Let $0 \neq x \in E(A)$ and $(x)^R$ is maximal in $S(A)$, Let $\lambda \in \Lambda = \mathrm{Hom}(E(A), E(A))$ such that $\lambda(A) = 0$. Since $Rx \neq 0$, there is $r' \in R$ such that $0 \neq r'x$ and $r'x \in A$ and hence $r'\lambda(x) = 0$. Now $(x)^R \subseteq (\lambda(x))^R$ and $r' \notin (x)^R$. Since this contradicts the maximality of $(x)^R$ in $S(A)$, we conclude that if $\lambda \in \Lambda$ and $\lambda(A) = 0$. Then $\lambda(x) = 0$ and thus $x \in C(A) = \cap \{\mathrm{Ker}\lambda \mid \lambda(A) = 0\}$.

THEOREM 3.7. If R is left subcommutative, then every simple R-module is rationally complete.

Proof. Let A be a left R-module and let $x \in C(A)$. For $a \in A$, $A = Ra \subseteq' Rx$. Let $M = (a)^R$ and $I = (x)^R$. Then $(a)^R$ is a maximal left ideal of R. Let ϕ be the mapping from R/M into R/I by the composition $R/M \longrightarrow A \longrightarrow Rx \longrightarrow R/I$. Let $\phi(1+M) = y + I$.

Then $\phi(i+M) = i\phi(1+M) = iy + I = 0 + I = 0$. But $\mathrm{Ker}\phi = M$, so that $I \subseteq M$.

Hence $(x)^R \subseteq (a)^R$. Since $x \in C(A)$, $(x)^R = (a)^R$. Thus $(x)^R$ is a maximal left ideal of R, so that Rx is a simple left module and thus $Rx = A$.

THEOREM 3.8. If R is a left perfect D ring, then every R-module is rationally complete.

Proof. Suppose that R is a left perfect D ring. By theorem 2.8, R is right perfect and left artinian and by proposition 2.6, R is left subcommutative. Thus by theorem 3.7, every left simple module over R is rationally complete. By [5] every left R-module is rationally complete.

References

1. H. Bass, Finitistic dimension and homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95(1960), 466-488.
2. J. Lambek, Lectures on rings and Modules, Waltham: Blaisdell 1966.
3. Wolfgang Schwartz, Über die Lösbarkeit gewisser ungleichhungen durch Primzhahlen, ibid. 212 (1963) 150-157.
4. W. Stephenson, Modules whose lattice of submodules is distributive. J. London Math. Soc. 28 (1974) 291-310.
5. H. Storrer, Rational extensions of modules, Pacific. J, Math. 38. no. 3 (1971) 785-794.