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A Note on Metrization of Topological Spaces

by

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A.H. Frink [2] proved that: Let X be a semimetrizable space via semimetric d such that $d(x, x_n)\to 0$ whenever $d(x, y_n)\to 0$ and $d(x_n, y_n)\to 0$, then X is metrizable. Using this fact, we have the following metrization theorem for semidevelopable spaces and Nagata spaces.

Theorem. For a topological space X, the following are equivalent;

(1) X is metrizable.

(2) X is semimetrizable by a semimetric d such that $d(x, y_n)\to 0$ and $d(x_n, y_n)\to 0$ imply $d(x, x_n)\to 0$,

(3) X is T_0 and has a semidevelopment $\gamma=(\gamma_1, \gamma_2, \gamma_3\cdots\cdots\cdots)$ such that $\{st^2(x, \gamma_n) : n\in N\}$ is a local base at x, and

(4) X has a decreasing Nagata structure $\{S_n(x)\}$, $\{U_n(x)\}$ such that $x\in S_n(y)$ iff $y\in S_n(x)$.

Proof. ($1\Leftrightarrow 2$). See [2, p. 137]

($2\Rightarrow 3$). Let $\gamma_n=\{A :$ diameter of $A<1/n\}$ for each n. Then $st(x, \gamma_n)=S(x;1/n)$, $1/n$ -sphere centered at x, and hence $\gamma=(\gamma_1, \gamma_2, \gamma_3, \cdots\cdots\cdots)$ is a semidevelopment for X.

Assume that there is an open set U containing x such that there exists an $x_n\in st^2(x, \gamma_n)-U$ for each $n\in N$. Then there exists a sequence $\{y_n\}$ such that $d(x, y_n)<1/n$ and $d(y_n, x_n)<1/n$. These imply that the sequence $\{x_n\}$ converges to x, which is a contradiction.

($3\Rightarrow 4$). Let $S_n(x)=st(x, \gamma_n)$ and $U_n(x)=st^2(x, \gamma_n)$. If $S_n(x)\cap S_n(y)\neq\phi$, then $x\in st^2(y, \gamma_n)=U_n(y)$. This shows that $\{S_n(x)\}$ and $\{U_n(x)\}$ is a Nagata structure for X. Now it is clear that $x\in S_n(y)$ iff $y\in S_n(x)$.

($4\Rightarrow 2$). Define $d(x, y)=1/\inf\{j\in N : y\notin S_j(x)\}$. Then, $x\in\overline{M}$; iff $S_n(x)\cap M\neq\phi$ for each $n\in N$; iff there exists $x_n\in M$ and $d(x, x_n)<1/n$ for each $n\in N$; iff $d(x, M)=0$. This implies that d is a semimetric for X.

Now assume $d(x, y_n)\to 0$ and $d(y_n, x_n)\to 0$. For each $k\in N$, there exists an $n_0\in N$ such that $d(x, y_n)<1/k$ and $d(x_n, y_n)<1/k$ for every $n\geq n_0$. This implies that $S_k(x)\cap S_k(x_n)\neq\phi$, and hence the sequence $\{x_n\}$ converges to x. Therefore $d(x, x_n)\to 0$. This completes the proof.

References

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