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WEAK-AND STRONG CONVERGENCE IN SPACE OF LINEAR OPERATORS

by

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1. INTRODUCTION

Let E be a topological Vector space, and E' its topological dual. The topology on E associated with the family of continuous linear forms $f \in E'$ is called the weak topology of E. If E is a normed vector space then let us call the topology on E associated with the norm the strong topology. A sequence $\{x_n\}$ in a normed linear space E is said to be converges weakly (strongly) to x in E if $\{x_n\}$ converges to x provided E is equipped with the weak (strong) topology.

Choquet [1] proved that if E be a Hilbert space then the statement $\{x_n\}$ converges strongly to x is equivalent to $\{x_n\}$ converges weakly to x and $\|x_n\|$ tends to $\|x\|$.

Let E and F are normed linear spaces and $L(E, F)$ be the set of all continuous linear map T on E into F. The purpose of this paper is to show that the following two statements are equivalent whenever F is Hilbert space:

1) $\{T_n\}$ in $L(E, F)$ converges to T strongly, that is, for each $x \in E$, $\{T_n x\}$ converges to Tx strongly.
2) $\{T_n x\}$ converges to T weakly, that is, for each $x \in E$, $\{T_n x\}$ converges to Tx weakly and $\|T_n x\|$ tends to $\|Tx\|$.

2. PRELIMINARIES

Let a be a fixed element of a Banach space E. Let $\varepsilon > 0$ and let $f_1\ f_2 \cdots\cdots f_n$ be elements of E'. Consider the set

$$U(a) = U(a : f_1 f_2 \cdots\cdots f_n, \varepsilon) = \{x : |f_i x - f_i a| < \varepsilon\},\ i = 1.2, \cdots\cdots n$$

since the class of all sets $U(a)$ with a fixed is a base of neighborhoods of a in the weak topology on E, $\{x_n\}$ converges weakly to x is equivalent to, for every $f \in E'$, we have $f(x_n)$ converges to $f(x)$. On the other hand, $\{x_n\}$ converges strongly to x in E means that $\|x_n - x\|$ tends to zero.

LEMMA 1 : In a Hilbert space E. A sequence $\{x_n\}$ weakly conyerges to x if and only if $(x_n|y)$ converges to $(x|y)$ for each $y \in E$ where $(x|y)$ is the inner product of x and y.

Proof. Since the map $P_y : x \to (x|y)$ is a continuous linear form for eyery $y \in E$, the sequence $P_y(x_n)$ converges to $P_y(x)$, that is, $(x_n|y)$ converges to $(x|y)$ for each $y \in E$. On the other hand, from Frechet-Riesz theorem, for every $f \in E'$ there exists an element $y \in E$ such that $f(x) = (x|y)$ for all $x \in E$. It follows that by Hypothesis $(x_n|y)$ converges to $(x|y)$ implies $f(x_n)$ converges to $f(x)$ for every $f \in E'$. Hence $\{x_n\}$ converges weakly to x.

LEMMA 2 : In a Hilbert space E. If $\{x_n\}$ converges strongly to x then $\{x_n\}$ converges weakly to x.

Proof. Since $\|x_n - x\|$ tends to zero, for every $\varepsilon > 0$ and for every non zero y in E there exists a

$N>0$ such that $n>N$ implies $\| x_n - x \| < \frac{\varepsilon}{\| y \|}$, and now, by Cauchy Schwarz inequality $| (x_n | y) - (x | y) | \leq \| x_n - x \| \ \| y \| < \varepsilon$ provided $n>N$. Thus $(x_n | y)$ converges to $(x | y)$ for every $y \in E$. By Lemma 1, converges weakly to x.

The converse of Lemma 2 is not true. The sequence (e_n) of unit vectors in Hilbert space l^2 does not tend to zero since $\| e_n \| = 1$ for every n. It tends, however, to zero weakly. Indeed, $(e_n | a) = \bar{a}_n$ and $\bar{a}_n$ tends to zero because $\sum | a_n |^2$ converges implies $(a_n)_{n \in N}$ converges to zero, where $a = (a_n)$ is an element of l^2.

LEMMA 3. Let E be a Hilbert space. x a point of E and $\{x_n\}$ a sequence of E : the following statements are equivalent.

1) $\{x_n\}$ converges strongly to x.

2) $\{x_n\}$ converges weakly to x and $\| x_n \|$ tends to $\| x \|$.

Proof. By Lemma 2 and $| \, \| x \| - \| x \| \, | < \| x_n - x \|$, 1)$\Rightarrow$2) is obvious. Let us prove the converse. Since weak convergence implies $(x_n | x)$ converges to $(x | x) = \| x \|^2$, $\| x_n - x \|^2 = \| x_n \|^2 + \| x \| - (x_n | x) - (x | x_n)$ tends to zero as $n \to \infty$. In other words $\| x_n - x \|$ tends to zero, that is, $\{x_n\}$ converges strongly to x.

3. THEOREMS

Let E and F be normed linear spaces. The linear space $L(E, F)$ of continuous linear mappings $T: E \to F$ have three topologies as the following definition.

Definition 1 : The Uniform operator topology in $L(E, F)$ is defined by the metric topology of $L(E, F)$ induced by its norm $\| T \| = \sup_{\| x \leq 1} \| Tx \|$. In $L(E, F)$ equipped with the uniform operator topology, a sequence $\{T_n\}$ converges to T means that $\| T_n - T \|$ tends to zero. In this case, we call $\{T_n\}$ uniformly converges to T.

Definition 2 : The family of sets $U(T : A, \varepsilon) = \{R \in L(E, F) : \| (T-R)x \| < \varepsilon, \ x \in A\}$, where A is a finite subset of E and $\varepsilon > 0$ is arbitrary, forms a basis of neighborhoods of a topology for $L(E, F)$. This topology is called the strong operator topology for $L(E, F)$. In the strong operator topology, a sequence $\{T_n\}$ converges to T in $L(E, F)$ if and only if $\{T_n x\}$ strongly converges to Tx for every $x \in E$ in F. We can say roughly that a sequence $\{T_n\}$ strongly converges to T if $\| T_n x - Tx \|$ tends to zero for every $x \in E$ in F.

Definition 3 : The weak operator topology in $L(E, F)$ is the topology defind by the basis of neighborhoods of T,

$$U(T : A, B, \varepsilon) = \{R \in L(E, F) : \| f(T-R)x \| < \varepsilon, \ f \in B, \ x \in A\}$$

where A and B are an arbitrary finite subset in E and F' (the dual of F) respectively. Thus, in the weak operator topology, a sequence $\{T_n\}$ converges to T means that $\{f T_n x\}$ converges fTx for every $x \in E$ and f in F' and also equivalent to, the sequence $\{T_n x\}$ weakly converges to Tx for each $x \in E$ in the scense of §2 in F. We say that a sequence $\{T_n\}$ weakly converges to T if $\{T_n\}$ converges to T in the weak operator topology for $L(E, F)$.

THEOREM 1 : Let E and F are normed linear spaces. In $L(E, F)$, if a sequence $\{T_n\}$ uniformly converges to T then $\{T_n\}$ strongly converges to T.

Proof: For every non zero element x of E and positive real ε, there exists N such that $n>N$ implies $\| T_n - T \| < \frac{\varepsilon}{\| x \|}$ because $\{T_n\}$ uniformly converges to T. From $\| T_n x - Tx \| \leq \| T_n - T \| \ \| x \|$ for every $x \in E$, we have $\| T_n - T \| < \varepsilon$ for $n>N$, for every $x \in E$. Hence $\{T_n\}$ strongly converge to T.

THEOREM 2 : Let E be a normed linear space and F a Hilbert space. A sequence $\{T_n\}$ in $L(E, F)$ weakly converges to T if and only if $\{(T_nx|y)\}$ tends to $(Tx|y)$ for each pair (x, y) in $E\times F$.

Proof: By Definition 3 $\{T_n\}$ weakly converges to T means that $\{T_nx\}$ weakly converges to Tx for each $x\in E$ in the Hilbert space F. Lemma 1 asserts that the theorem is true.

THEOREM 3 : Let E be a normed linear space and F a Hilbert space. If $\{T_n\}$ strongly converges to T then $\{T_n\}$ weakly converges to T.

Proof: By Cauchy Schwarz inequelity, for every pair (x, y) in $E\times F$ $|(T_nx|y)-(Tx|y)|$ $\|T_nx-Tx\|$.

Since $\{T_n\}$ strongly converges to T, $\|T_nx-Tx\|$ tends to zero. Thus, from theorem 2, $\{T_n\}$ converges to T weakly.

THEOREM 4 : Let E be a normed linear space and F a Hilbert space, and let T, T in $L(E, F)$. The following two statements are equivalent.

1) $\{T_n\}$ strongly converges to T.

2) $\{T_n\}$ weakly converges to T and $\|T_nx\|$ tends to $\|Tx\|$ for every $x\in E$.

Proof. From Lemma 3, Definition 2, 3 and Theorem 3, the Theorem is true.

Reference

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요 약

약 위상과 강위상이 주어진 선형공간 E에서 수열의 약수렴과 강수렴에 관한 성질을 보조정리로 하여 두 Banach 공간 E와 F사이에 정의된 연속작용소 공간 $L(E. F)$에서 작용소열 $\{T_n\}$이 T에 강수렴하기 위한 필요충분조건은 F가 Hilbert 공간일때는 $\{T_n\}$이 T에 약수렴함과 동시에 실수열 $\|T_nx\|$가 실수 $\|Tx\|$에 수렴할 때이다.