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On properties of locally convex Baire spaces

by

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1. Introduction

A subset A of a topological space E is a rare (nowhere dense) if the closure of E has a void interior, i.e., if $(\bar{A})^\circ=\phi$. Clearly A is rare in E if and only if $(\bar{A})^c$ is every-where dense in E. We say that the subset A of E is meager (of first category) if there exists a sequence (A_n) of rare subsets of E such that $E=\bigcup_{n=1}^{\infty} A_n$.

Finally, we say that a topological space E is a Baire space if no nonempty open subset of E is meager. If E is a Baire space then E itself is not the union of countably many rare subsets. Since the finite union of rare sets is rare, (the finite intersection of open, everywhere dense sets is open and everywhere dense) a topological space is a Baire space is a Baire space if and only if it can be written as the union of an increasing (nondecreasing) sequence of rare sets.

The following definitions are familiar and historically important notions. A locally convex space E

(1) has property (S) if its dual E' is $\sigma(E',E)$-sequentially complete;

(2) has property (C) if every $\sigma(E',E)$-bounded subset of E' is $\sigma(E',E)$-relatively compact;

(3) is barrelled if every $\sigma(E',E)$-bounded subset of E' is equicontinuous;

(4) is Baire-like if E is not the union of an increasing sequence of rare, balanced, convex sets;

(5) is Baire if E is not the union of (increasing) sequence of rare sets.

In this paper, we prove the inclusion relationships that a space of type (n) is necessarily of type $(n-1)$ for $n=5,4,3,2$.

2. Inclusion relations

Theorem 1. A locally convex space E which has Property(C) has property(S).

Proof: Let (x_n) be a $\sigma(E',E)$-Cauchy sequence. Then the set A of ponints x_n is $\sigma(E',E)$-bounded. Since E has property (C), set A is relatively countably compact and hence sequence (x_n) has $\sigma(E',E)$ cluster point x in E. But sequence (x_n) being a $\sigma(E',E)$-Cauchy sequence, sequence (x_n) weakly converges to x. Hence we complete proof.

Theorem 2. Barrelled space E has property (C).

Proof: Suppose E is barrelled space. Let A be a $\sigma(E',E)$- bounded subset of E. Then A is equicontinuous set. But equicontinuous set A being relatively $\sigma(E',E)$-relatively compact, A is relatively $\sigma(E',E)$-relatively compact. Since relatively compact is relatively countably compact, A is $\sigma(E',E)$-relatively countably compact. This proves that E has property(C).

Prodosition. E is a barrelled space if and only if there does not exist a rare, balanced, convex set A such that $E=\bigcup_{n\geq 1}(nA)$.

Proof: Suppose E be a barrelled space, and there exist a rare, balanced, convex set A such that $E=\bigcup_{n\geq 1}(nA)$. Then $E=\bigcup_{n\geq 1}(n\bar{A})$, this implies that $\bar{A}$ is absorbing set and hence $\bar{A}$ is barrel.

Since E is barrelled, $\bar{A}$ is a neighborhood of o in E. On the other hand, since A is a rare and so is $\bar{A}$, interior of $\bar{A}$ is empty. This gives a contradiction.

Conversely, let A be a barrel in E. Since A is A is absorbing, we have $\bigcup_{n\geq 1}(nA)=E$. By assumption, the set A has an interior point x_0. Let V be a balanced neighborhood of o in E such that $x_0+V\subset A$. Since A is balanced, we have $-x_0+V\subset A$. But then $V\subset A$ since if $x\in V$, then $x=1/2\ (x_0+x)+1/2(-x_0+x)\in A$, because A is convex. Hence A is a neighborhood of o in E. This completes proof.

Theorem 3. A locally convex Baire-like space E is barrelled.

Proof: Let A be a rare, balanced, convex set. Since $x\rightarrow nx$ is a homeomorphism, the set nA is a rare, balanced, convex set and $\{nA\}$ is an increasing sequence because A is balanced set.

Since E is Baire-like, E cannot be covered by sequence $\{nA\}$ of rare, balanced, convex sets. It f ollows from proposition that E is barrelled.

Theorem 4. A locally convex Baire space E is a locally convex Baire-like space.

Proof: It is obvious, in view of the definitions concerned.

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